Session 9
Mar. 2, 2020

recall: • V a topological vector space, V' its dual

• weak convergence $\underset{n\to\infty}{w\text{-}\lim} f_n = f \in V$ means $T(f_n) \to T(f)$ $\forall\, T \in V'$

• weak* - convergence $\underset{n\to\infty}{w^*\text{-}\lim} T_n = T$ means $T_n(f) \to T(f)$ $\forall\, f \in V$

next: extend $\mathcal{F}$ and ∂_x^α to operators $S' \to S'$

Theorem 2.30:

Let $A: S \to S$ be linear and continuous. Then the $\boxed{\text{adjoint } A': S' \to S'}$, defined via

$$\overbrace{\underbrace{(A'\underbrace{T}_{\in S'})}_{\in S'}\ (\underbrace{f}_{\in S})}^{\in \mathbb{C}} := \underbrace{T(Af)}_{=(f, A'T)_{S,S'} = (Af, T)_{S,S'}} \quad \forall\, f \in S,$$

is a weak* continuous linear map.

Proof: first, $A'T \in S'$, since $T \circ A$ composition of continuous maps

sequential continuity: Let $T_n \xrightarrow{*} T$, then $\forall\, f \in S$:

$$(A'T_n)(f) = T_n(Af) \xrightarrow{n\to\infty} T(Af) = (A'T)(f), \text{ so } A'T_n \xrightarrow{*} AT$$

problem: topology in S' not given by a metric, so sequential continuity does not necessarily imply continuity

but here it does, using the topological concept of nets (proof omitted) $\square$

Definition 2.31: $\mathcal{F}_{S'} = \mathcal{F}_S'$, meaning for $T \in S'$, we define its Fourier transform

$$\hat{T} \in S' \text{ by } \hat{T}(f) = T(\hat{f}) \quad \forall\, f \in S.$$

Corollary 2.32: $\mathcal{F}': \mathcal{S}' \to \mathcal{S}'$ is a weak*-continuous bijection, and $\hat{T}_f = T_{\hat{f}}$ for all $f \in \mathcal{S}$ (or $f \in L^1$) (recall $T_f(g) := \int fg$).

Proof: $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ is continuous and linear, so we conclude with Thm. 2.30 that $\mathcal{F}': \mathcal{S}' \to \mathcal{S}'$ is weak*-continuous.

Bijective? $(\mathcal{F}^{-1'}\mathcal{F}'T)(f) = (\mathcal{F}'T)(\mathcal{F}^{-1}f) = T(\mathcal{F}\mathcal{F}^{-1}f) = T(f)$

$\Rightarrow$ yes, with continuous inverse $\mathcal{F}'^{-1} = \mathcal{F}^{-1'}$.

Also, for $f \in \mathcal{S}$ or $f \in L^1$:

$$\hat{T}_f(g) = (\mathcal{F}T_f)(g) = T_f(\mathcal{F}g) = \int f(x)\,\hat{g}(x)\,dx \overset{\text{Plancherel}}{=} \int \hat{f}(x)g(x) = T_{\hat{f}}(g) \quad \forall g \in \mathcal{S} \qquad \square$$

Ex.: Fourier transform of δ ($\delta(f) = f(0)$)

$$\Rightarrow \hat{\delta}(f) = \delta(\hat{f}) = \hat{f}(0) = \int \underbrace{(2\pi)^{-\frac{d}{2}}}_{g(x)} f(x)\,dx = T_g(f)$$

$\Rightarrow T_g$ with $g(x) = (2\pi)^{-\frac{d}{2}}$ is the Fourier transform of δ, or "$\hat{\delta}(k) = (2\pi)^{-\frac{d}{2}}$"

next: derivatives

note: $\partial_x^\alpha : \mathcal{S} \to \mathcal{S}$ is linear (clear) and continuous, since

$$\|\partial_x^\alpha f\|_{\gamma,\beta} = \|x^\gamma \partial_x^\beta \partial_x^\alpha f\|_\infty = \|f\|_{\gamma,\alpha+\beta}$$ (i.e., continuity on $\mathcal{S}$ follows as usual from sequential continuity)

Definition 2.34: $\partial_x^{\alpha\,\prime} = \left((-1)^{|\alpha|}\partial_x^\alpha\right)' : \mathcal{S}' \to \mathcal{S}'$, i.e., for $T \in \mathcal{S}'$ the distributional derivative $\partial_x^{\alpha\,\prime} T$ is defined by $(\partial_x^{\alpha\,\prime} T)(f) := T\left((-1)^{|\alpha|}\partial_x^\alpha f\right)$ $\forall f \in \mathcal{S}$.

Corollary 2.35: $\tilde{\partial}_x^\alpha : \mathcal{S}' \to \mathcal{S}'$ is weak*-continuous and $\tilde{\partial}_x^\alpha T_g = T_{\partial_x^\alpha g}$ $\forall g \in \mathcal{S}$.

Proof: weak*-continuity follows again from Thm. 2.30.

Also, $(\tilde{\partial}_x^\alpha T_g)(f) = T_g\left((-1)^{|\alpha|}\partial_x^\alpha f\right) = \int g(x)(-1)^{|\alpha|}\partial_x^\alpha f(x)\,dx$

|α| times integration by parts

$$= \int \left(\partial_x^\alpha g(x)\right) f(x)\,dx = T_{\partial_x^\alpha g}(f) \quad \forall f \in \mathcal{S}.$$

Ex.:
- for $\Theta(x) = \mathbb{1}_{[0,\infty)}(x)$, we find $\frac{d}{dx}\Theta = \delta$, see HW.
- $\partial_x^\alpha \delta$? See HW.

Corollary 2.38: For $g \in \mathcal{S}$, $(g * T)(f) = T(\tilde{g} * f)$ with $\tilde{g}(x) = g(-x)$ defines a weak*-continuous map, and $g * T_h = T_{g*h}$ for $h \in \mathcal{S}$.

Corollary 2.37: For $g \in C^\infty_{pol}$, $(gT)(f) = T(gf)$ defines again a weak*-continuous map

Proofs: similar to before.

Remarks:

- gT well-defined for $g \in C^\infty_{pol}$, but product of distributions undefined (much research effort to define it at least for some distributions, e.g., Hairer's regularity structures).

- $\{T_f \in S' : f \in S\}$ is dense in S' w.r.t. weak*-topology (not obvious, proof omitted). (T_f allows us to identify S with some subset of S'.)

  Thus, for $A: S \to S$ continuous and linear, the definition $A' T_f = T_{Af}$ uniquely defines A'! So it makes sense to say that $A' T_f := T_{Af}$ uniquely extends $A: S \to S$ to an operator $S' \to S'$ and just write $A' \equiv A$.

$\boxed{\text{Conclusion: We have defined } \mathcal{F}T,\ \partial_x^\alpha T,\ gT \text{ (for } g \in C^\infty_{pol}),\ g * T \text{ (for } g \in S) \text{ on } S'.}$