Session 10
Mar. 4, 2020

last time: • adjoints $A': S' \to S'$, $(A'T)(f) = T(Af)$ or $(f, A'T)_{S,S'} = (Af, T)_{S,S'}$

↳ weak*-continuous

• adjoints of $\mathcal{F}$, ∂_x^α, M_f, $*$

• extend A to S' by $AT_f = T_{Af}$, $\{T_f \in S' : f \in S\}$ dense in S'

Application to SE:

Theorem 2.40:

Let $\psi_0 \in S'$, then the unique global solution to the free Schrödinger equation

$i\partial_t \psi = -\frac{1}{2}\Delta\psi$ (in the sense of distributions) with $\psi(0) = \psi_0$ is $\psi(t) = \mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F}\psi_0$,

with $\psi \in C^\infty(\mathbb{R}_t, S'(\mathbb{R}^d))$.

Proof: First, note that $\psi(t) = \mathcal{F}^{-1} e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0 \in S'$ due to 2.32 and 2.37.

Next, let us check if this $\psi(t)$ solves the SE. For any $f \in S$, we find

$$i\frac{d}{dt}(f, \psi(t))_{S,S'} = i\frac{d}{dt}\left(f, \mathcal{F}^{-1} e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0\right)_{S,S'}$$

$$\overset{\text{by def.}}{=} i\frac{d}{dt}\left(\mathcal{F} e^{-i\frac{k^2}{2}t}\mathcal{F}^{-1} f, \psi_0\right)_{S,S'}$$

$$\overset{\text{continuity}}{=} \left(\mathcal{F}\left(i\frac{d}{dt} e^{-i\frac{k^2}{2}t}\right)\mathcal{F}^{-1} f, \psi_0\right)_{S,S'}$$

$$= \left(\mathcal{F} e^{-i\frac{k^2}{2}t} \underbrace{\frac{k^2}{2}\mathcal{F}^{-1} f}_{=\mathcal{F}^{-1}(-\frac{\Delta}{2}f)}, \psi_0\right)_{S,S'}$$

$$= \left(-\frac{\Delta}{2} f, \mathcal{F}^{-1} e^{-i\frac{k^2}{2}} \mathcal{F} \psi_0\right)_{S,S'}$$

$$= \left(f, -\frac{\Delta}{2}\psi(t)\right)_{S,S'}$$

similarly $\left(i\frac{d}{dt}\right)^k (f,\psi(t))_{S,S'} = \left(\left(-\frac{\Delta}{2}\right)^k f, \psi(t)\right)_{S,S'}$, so $\psi(t) \in C^\infty(\mathbb{R}_t, S'(\mathbb{R}^d))$. $\square$

2.3 Long-time Asymptotics and the Momentum Operator

recall: probability that particle at time t is in $\Lambda \subset \mathbb{R}^d$ is $\mathbb{P}(X(t)\in\Lambda) = \int_\Lambda |\psi(t,x)|^2 dx$

What about momentum (=velocity here, since mass $m=1$)? A-priori not defined in QM.

We consider the asymptotic velocity $= \frac{\text{distance}}{\text{time}}$ for large times t

(see discussion later for what "large t" means).

Probability that velocity $\in \Lambda \subset \mathbb{R}^d$ is $\mathbb{P}\left(\frac{X(t)}{t}\in\Lambda\right) = \mathbb{P}(X(t)\in t\Lambda) = \int_{t\Lambda} |\psi(t,x)|^2 dx$

Next: find an expression for $\lim_{t\to\infty} \mathbb{P}\left(\frac{X(t)}{t}\in\Lambda\right)$ for $\psi(t,x) = (2\pi i t)^{-\frac{d}{2}} \int e^{i\frac{(x-y)^2}{2t}} \psi_0(y)\,dy$

Lemma 2.41:

For $\psi_0 \in S$, the solution to the free Schrödinger equation is

$$\psi(t,x) = \frac{e^{i\frac{x^2}{2t}}}{(it)^{d/2}} \hat{\psi}_0\left(\frac{x}{t}\right) + r(t,x) \qquad \text{with } \lim_{t\to\infty} \|r(t,\cdot)\|_{L^2} = 0$$

Proof: $\psi(t,x) = (2\pi i t)^{-\frac{d}{2}} \int e^{i\frac{(x-y)^2}{2t}} \psi_0(y)dy$

$$= \frac{e^{i\frac{x^2}{2t}}}{(it)^{d/2}} (2\pi)^{-\frac{d}{2}} \int e^{-i\frac{x}{t}y} \left(e^{i\frac{y^2}{2t}} - 1 + 1\right) \psi_0(y)dy$$

$$= \frac{e^{i\frac{x^2}{2t}}}{(it)^{d/2}} \left(\hat{\psi}_0\left(\frac{x}{t}\right) + \hat{h}_t\left(\frac{x}{t}\right) \right), \text{where } h_t(y) = \left(e^{i\frac{y^2}{2t}} - 1\right)\psi_0(y)$$

i.e., $r(t,x) = \frac{e^{i\frac{x^2}{2t}}}{(it)^{d/2}} \hat{h}_t\left(\frac{x}{t}\right)$

$$\|r(t,\cdot)\|_{L^2}^2 = \int |r(t,x)|^2 dx = t^{-d} \int |\hat{h}_t(\tfrac{x}{t})|^2 dx \overset{\frac{x}{t}=y}{=} \int |\hat{h}_t(y)|^2 dy$$

$$\overset{\text{Plancherel}}{=} \int |h_t(y)|^2 dy = \int \underbrace{\underbrace{|e^{i\frac{y^2}{2t}} - 1|^2}_{\substack{\to 0 \text{ as } t\to\infty \\ \forall y \in \mathbb{R}^d}} |\psi_0(y)|^2}_{\leq 4|\psi_0(y)|^2 \in L^1(\mathbb{R}^d)} dy \xrightarrow{t\to\infty} 0 \text{ by dominated convergence}$$

$\square$

Theorem 2.42:

Let $\psi(t,x)$ be the solution to the free SE with initial condition $\psi_0 \in S$, let $\Lambda \subset \mathbb{R}^d$ be measurable. Then

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{X(t)}{t} \in \Lambda\right) := \lim_{t\to\infty} \int_{t\Lambda} |\psi(t,x)|^2 dx = \int_{\Lambda} |\hat{\psi}_0(p)|^2 dp.$$

Proof: With Lemma 2.41 we find

$$\int_{t\Lambda} |\psi(t,x)|^2 dx = \underbrace{t^{-d} \int_{t\Lambda} |\hat{\psi}_0(\tfrac{x}{t})|^2 dx}_{= \int_{\Lambda} |\hat{\psi}_0(p)|^2 dp} + R(t)$$

$$\text{with } \lim_{t\to\infty} R(t) = \underbrace{\lim_{t\to\infty} \int_{t\Lambda} |r(t,x)|^2 dx}_{= 0 \text{ with Lemma 2.41}} + \lim_{t\to\infty} \underbrace{2 \operatorname{Re} t^{-\frac{d}{2}} \int_{t\Lambda} \overline{\hat{\varphi}_0\left(\tfrac{x}{t}\right)} \hat{h}_t\left(\tfrac{x}{t}\right) dx}_{}$$

$$= 2 \operatorname{Re} \int_{\Lambda} \overline{\hat{\varphi}_0(p)} \hat{h}_t(p) dp$$

Cauchy-Schwarz

$$\leq 2 \, \|\hat{\varphi}_0\|_{L^2} \underbrace{\|\hat{h}_t\|_{L^2}}_{}$$

$$= \|r(t,\cdot)\|_{L^2} \xrightarrow{t\to\infty} 0 \text{ by Lemma 2.41}$$

$\square$