Session 11
Mar. 9, 2020

Last time: • $\psi(t,x)$ solution to free SE, $\psi_0 \in \mathcal{S}$, then

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{X(t)}{t} \in \Lambda\right) := \lim_{t\to\infty} \int_{t\Lambda} |\psi(t,x)|^2 dx = \int_\Lambda |\hat{\psi}_0(p)|^2 dp.$$

Remarks:

• $|\hat{\psi}(t,p)|^2 = |e^{-i\frac{p^2}{2}t}\hat{\psi}_0(p)|^2 = |\hat{\psi}_0(p)|^2 = |e^{iap}\hat{\psi}_0(p)|^2$, so result is independent from choice of $t=0$ or $x=0$

• expectation value of asymptotic momentum:

$$\mathbb{E} = \int p\,|\hat{\psi}_0(p)|^2 dp = \int \overline{\hat{\psi}_0(p)}\, p\, \hat{\psi}_0(p)\,dp = \int \overline{\psi(t,x)}\,(-i\nabla)\,\psi(t,x)\,dx$$

$$= \langle \psi_t, P\psi_t\rangle \quad \text{where } P = -i\nabla_x \text{ is called "momentum operator"}$$

Different viewpoint:

Consider macroscopic scales $\frac{x}{\varepsilon}, \frac{t}{\varepsilon}$ for small ε

$\Rightarrow$ def. $\psi_\varepsilon(t,x) = \varepsilon^{-\frac{d}{2}}\psi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)$, s.t. $\|\psi_\varepsilon(t,\cdot)\|_{L^2} = \|\psi(\frac{t}{\varepsilon},\cdot)\|_{L^2} = \|\psi_0\|_{L^2}$

$$\Rightarrow i\partial_t\psi_\varepsilon(t,x) = \varepsilon^{-\frac{d}{2}}\underbrace{i\frac{\partial\psi}{\partial t}\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)}_{\substack{= -\frac{1}{2}(\Delta\psi)(\frac{t}{\varepsilon},\frac{x}{\varepsilon}) \\ = -\frac{1}{2}\varepsilon^2\Delta_x\psi(\frac{t}{\varepsilon},\frac{x}{\varepsilon})}}\frac{1}{\varepsilon} = \varepsilon\left(-\frac{1}{2}\Delta_x\right)\psi_\varepsilon(t,x)$$

$$\Rightarrow i\varepsilon\partial_t\psi_\varepsilon(t,x) = -\frac{\varepsilon^2}{2}\Delta_x\psi_\varepsilon(t,x)$$

Note: • recall that in SI units the SE is $i\hbar\partial_t \psi(t,x) = -\frac{\hbar^2}{2m}\Delta_x \psi(t,x)$, so formally

$\lim_{\varepsilon\to 0}$ is the same as $\lim_{\hbar\to 0}$ (which people often consider although $\hbar$ is a physical constant)

Lemma 2.41 then says that $$\psi_\varepsilon(t,x) = \frac{e^{i\frac{x^2}{2\varepsilon t}}}{(it)^{d/2}}\hat{\psi}_0\left(\frac{x}{t}\right) + \underbrace{\varepsilon^{-\frac{d}{2}} r\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)}_{=: r_\varepsilon(t,x)}$$

with $\|r_\varepsilon\|^2 = \varepsilon^{-d}\int |r(\frac{t}{\varepsilon},\frac{x}{\varepsilon})|^2 dx$

$= \int |r(\frac{t}{\varepsilon},y)|^2 dy \xrightarrow{\varepsilon\to 0} 0$

i.e., $|\hat{\psi}_0(p)|^2$ is the asymptotic momentum distribution as $\varepsilon\to 0$

A more direct connection to velocity:

consider the probability density $\rho_\psi(t,x) = |\psi(t,x)|^2$

$$\Rightarrow \partial_t\rho_\psi(t,x) = \partial_t|\psi(t,x)|^2 = i\,\overline{(i\partial_t\psi(t,x))}\,\psi(t,x) - i\,\overline{\psi(t,x)}\,(i\partial_t\psi(t,x))$$

$$= \frac{i}{2}\,\overline{(-\Delta\psi(t,x))}\,\psi(t,x) - \frac{i}{2}\,\overline{\psi(t,x)}\,(-\Delta\psi(t,x))$$

$\frac{i}{2}z - \frac{i}{2}\bar{z} = -\operatorname{Im} z$

$$= -\operatorname{Im}\overline{\psi(t,x)}(\Delta\psi(t,x))$$

$$= -\nabla\underbrace{\operatorname{Im}\overline{\psi(t,x)}(\nabla\psi(t,x))}_{=: j_\psi(t,x) = \text{current}} \qquad (\text{since } \overline{\nabla\psi}\,\nabla\psi \in \mathbb{R})$$

$\Rightarrow$ $\boxed{\partial_t\rho_\psi + \nabla j_\psi = 0 \text{, continuity equation}}$ (Note: also holds if $i\partial_t\psi = -\frac{\Delta}{2}\psi + V\psi$ with $V(x)\in\mathbb{R}$)