Session 14
March 18, 2020

Today: operators between normed spaces

(motivation should be clear, for us it is, e.g., Hamiltonians or propagators)

In ∞-dimensions, there are not just bounded, but also unbounded operators.

Definition 3.16: Let X and Y be normed spaces. A linear operator $L: X \to Y$ is bounded if $\exists C < \infty$ with $\underbrace{\|Lx\|_Y}_{\text{norm on } Y} \leq C \underbrace{\|x\|_X}_{\text{norm on } X}$ $\forall x \in X$.

Proposition 3.17: The space $\mathcal{L}(X,Y) = \{L: X \to Y, L \text{ linear and bounded}\}$ with norm $\boxed{\|L\|_{\mathcal{L}(X,Y)} := \sup_{\substack{x \in X \\ \|x\|_X = 1}} \|Lx\|_Y}$ is itself a normed space.

$\underbrace{\text{If } Y \text{ is a Banach space}}_{\text{not necessarily } X}$, also $\mathcal{L}(X,Y)$ is a Banach space.

Proof: HW 5

Why are bounded operators so interesting? Because these are also the continuous ones!
(And since we deal with linear ones, it is enough to check continuity at 0.)

Lemma 3.18: Let $L: X \to Y$ be linear (X, Y normed spaces). Then the following statements are equivalent:
(i) L is continuous at 0.
(ii) L is continuous.
(iii) L is bounded.

Proof: (iii) $\Rightarrow$ (i): Let $\|x_n\|_X \to 0 \Rightarrow \|Lx_n\|_Y \underset{\uparrow}{\leq} \|L\|_{\mathcal{B}} \|x_n\|_X \to 0$

L bounded

(i) $\Rightarrow$ (ii): Let $\|x_n - x\|_X \to 0 \Rightarrow \|Lx_n - Lx\|_Y \overset{\text{linearity}}{=} \|L(x_n - x)\|_Y \xrightarrow{\text{continuity at } 0} 0$

(ii) $\Rightarrow$ (iii): suppose L not bounded, then $\exists$ a sequence $(x_n)_n$ with $\|x_n\|_X = 1$ and $\|Lx_n\|_X \geq n$ $\forall n \in \mathbb{N}$. Defining $z_n := \frac{x_n}{\|Lx_n\|_Y}$, then $\|z_n\| = \frac{\|x_n\|_X}{\|Lx_n\|_Y} \leq \frac{1}{n}$, i.e., $z_n \to 0$. But $\|Lz_n\|_Y = \frac{\|Lx_n\|_Y}{\|Lx_n\|_Y} = 1$, which contradicts continuity (at 0). $\square$

What do unbounded operators look like? Much more later, here just two examples:

- Define $\ell_0 = \{ (x_n)_n \in \ell^1 : \exists N \in \mathbb{N} \text{ s.t. } x_n = 0 \ \forall n \geq N \}$ with the norm

$$\|(x_n)_n\|_{\ell^1} = \underbrace{\sum_{n=1}^{\infty} |x_n|}_{\text{actually just a finite sum}}. \text{ Define } T: \ell_0 \to \ell_0,\ x \mapsto Tx = (x_1, 2x_2, 3x_3, \dots).$$

But if $(e^{(n)}_k)_k$ is the sequence with $e^{(n)}_k = \begin{cases} 1 & \text{for } k = n \\ 0 & \text{otherwise} \end{cases}$, in particular $\|e^{(n)}\| = 1$,

then $\|Te^{(n)}\| = n$, i.e., T is unbounded.

- Asymptotic momentum operator $-i\frac{d}{dx}$ on L^2. Clearly for $\psi \in L^2$, $-i\frac{d}{dx}\psi$ need not be in L^2.

(E.g., $L^2([0,1])$, then $f(x) = x^{-\frac{1}{4}} \in L^2([0,1])$ since $\int_0^1 |x^{-\frac{1}{4}}|^2 dx = \int_0^1 x^{-\frac{1}{2}} dx = 2x^{\frac{1}{2}}\Big|_0^1 = 2$.
But $-i\frac{d}{dx} f(x) = -i(-\frac{1}{4})x^{-\frac{5}{4}} \notin L^2([0,1])$ since $\int_0^1 |x^{-\frac{5}{4}}|^2 dx = \int_0^1 x^{-\frac{5}{2}} dx = -\frac{2}{3}x^{-\frac{3}{2}}\Big|_0^1$ does not exist.)

In the last chapter, we defined operators on S' by defining them on a dense subset and extending them by continuity. This can also be done here (fully rigorous):

Theorem 3.20: Let Z be a dense subspace of a normed space X, and let Y be a Banach space. Let $L: Z \to Y$ be a linear bounded operator. Then L has a unique linear bounded extension $\tilde{L}: X \to Y$ with $\underbrace{\tilde{L}|_Z = L}_{\tilde{L} \text{ and } L \text{ coincide on } Z \text{ of course}}$ and $\|\tilde{L}\|_{\mathcal{B}(X,Y)} = \|L\|_{\mathcal{B}(Z,Y)}$.

<u>Proof:</u> The idea should be clear: using continuity we "fill in the gaps."

Choose some $x \in X$, then $\exists$ sequence $(z_n)_n$ in Z with $\|z_n - x\|_X \to 0$

(using just density of Z in X; note: $x \in X$ is fixed, no completeness necessary).

$\Rightarrow (z_n)_n$ converges $\Rightarrow (z_n)_n$ is a Cauchy sequence.

$\Rightarrow \|Lz_n - Lz_m\|_Y \overset{\text{linearity}}{=} \|L(z_n - z_m)\|_Y \leq \|L\|_{\mathcal{L}(Z,Y)} \|z_n - z_m\|_Z$, i.e., also $(Lz_n)_n$ is a

Cauchy sequence in Y. Since Y is complete, $Lz_n \to y \in Y$.

But is this y independent of the choice of sequence?

Yes: if $\|z_n' - x\|_X \to 0$, also the sequence $(z_1, z_1', z_2, z_2', z_3, z_3', \ldots)$ converges to x and

as above $(Lz_1, Lz_1', Lz_2, Lz_2', \ldots)$ converges to some $\tilde{y} \in Y$. But every subsequence of a

convergent sequence converges to the same limit.

So we def. $\tilde{L}x := y$ with this construction.

$\|\tilde{L}\|_{\mathcal{L}(X,Y)} \leq \|L\|_{\mathcal{L}(Z,Y)}$

(and $\|L\|_{\mathcal{L}(Z,Y)} \leq \|\tilde{L}\|_{\mathcal{L}(X,Y)}$ clear by def.)

↳ linearity clear

↳ boundedness: $\|\tilde{L}x\|_Y = \lim_{n\to\infty} \underbrace{\|Lz_n\|_Y}_{\leq \|L\|_{\mathcal{L}(Z,Y)} \|z_n\|_Z} \leq \|L\|_{\mathcal{L}(Z,Y)} \|x\|_X \Rightarrow \tilde{L}$ continuous

and continuity on a dense subset implies that this is the unique extension. $\square$

Now, e.g., extension of the Fourier transform from $\mathcal{S}$ to L^2 follows as a simple corollary.

Let us first note:

$$\boxed{\underline{\text{Theorem 3.21:}}\ C_0^\infty(\mathbb{R}^d) \text{ is dense in } L^p(\mathbb{R}^d),\ 1 \leq p < \infty.}$$

Proof: From HW3, Problem 3(b), we know that C_0^∞ is dense in C_0 $\underbrace{\text{w.r.t. } \|\cdot\|_{L^p}}$.

density is defined w.r.t. a norm, or generally a topology (a subset might be dense w.r.t. one norm, but not another)

(We used convolution there to "smoothen out" (or "mollify") $f \in L^p$.)

It is also a standard result that C_0 is dense in L^p, which implies that C_0^∞ is dense in L^p. $\square$

Then we have

Theorem 3.22: The Fourier transform $\mathcal{F}: (S(\mathbb{R}^d), \|\cdot\|_{L^2(\mathbb{R}^d)}) \to L^2(\mathbb{R}^d)$ can be uniquely extended to a bounded linear operator $L^2 \to L^2$.

Furthermore:
- $\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2} \quad \forall f \in L^2$
- $\mathcal{F}\mathcal{F}^{-1} = \mathcal{F}^{-1}\mathcal{F} = \mathrm{id}_{L^2}$
- $(\mathcal{F}f)(k) = \underbrace{\lim_{N\to\infty}}_{L^2 \text{ limit, not pointwise}} (2\pi)^{-\frac{d}{2}} \int_{|x|<N} e^{-ikx} f(x)\,dx \quad \forall f \in L^2.$

Proof: $C_0^\infty \subset S \subset L^2$, so with Thm. 3.21 also S is dense in L^2 and we can apply Thm. 3.20.

Also: $\mathcal{F}\mathcal{F}^{-1}|_S = \mathcal{F}^{-1}\mathcal{F}|_S = \mathrm{id}_{L^2}|_S$, but since $\mathcal{F}, \mathcal{F}^{-1}, \mathrm{id}$ continuous, equality holds on L^2.

Limit formula follows directly from $\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2}$: let us denote $f_N(x) = f(x)\underbrace{\mathbb{1}_{B_N(0)}(x)}_{= \begin{cases} 1 & \text{for } |x|<N \\ 0 & \text{else} \end{cases}}$. Then $\lim_{N\to\infty} \|\mathcal{F}f - \mathcal{F}f_N\| = \lim_{N\to\infty} \|f - f_N\| = 0$. $\square$

Note:
- one can of course use any other suitable limit formula for explicit computations.
- so even for functions $\notin L^1$, we have defined $\int f(x) e^{-ikx}\,dx$.