Session 15
March 23, 2020

The properties of the Fourier transform on L^2 should remind you of a class of operators from linear Algebra:

$\boxed{\text{Definition 3.23: Let } \mathcal{H}_1, \mathcal{H}_2 \text{ be Hilbert spaces. A linear bounded operator } U \in \mathcal{L}(\mathcal{H}_1,\mathcal{H}_2) \text{ is called unitary if it is surjective and isometric (isometric meaning } \|U\psi\|_{\mathcal{H}_2} = \|\psi\|_{\mathcal{H}_1}\ \forall \psi\in\mathcal{H}_1).}$

Note: • injective follows from $\|U\psi\|_{\mathcal{H}_2} = \|\psi\|_{\mathcal{H}_1}$

• with the polarization identity isometry $\Leftrightarrow$ preservation of inner product:

$$\langle U\psi, U\varphi\rangle_{\mathcal{H}_2} = \langle \psi,\varphi\rangle_{\mathcal{H}_1} \quad \forall \psi,\varphi\in\mathcal{H}_1$$

• $\mathcal{F}: L^2 \to L^2$ is unitary

Let us now come back to the free Schrödinger equation.

We can define the free propagator on L^2 now:

for any (fixed) $t\in\mathbb{R}$: $P_f(t): L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$, $P_f(t) = \mathcal{F}^{-1} e^{-i\frac{k^2}{2}t}\mathcal{F}$.

$\Rightarrow P_f(t)$ is clearly unitary ($|e^{-i\frac{k^2}{2}t}| = 1$ and $\mathcal{F}$ isometric) for any $t\in\mathbb{R}$.

Is $\psi: \mathbb{R}\to L^2(\mathbb{R}^d)$, $t\mapsto \psi(t) = P_f(t)\psi_0$ continuous?

We check $\|\psi(t+h)-\psi(t)\|_{L^2}^2 = \|P_f(t+h)\psi_0 - P_f(t)\psi_0\|_{L^2}^2$

$$= \|\mathcal{F}^{-1}(e^{-i\frac{k^2}{2}t}e^{-i\frac{k^2}{2}h} - e^{-i\frac{k^2}{2}t})\overbrace{\mathcal{F}\psi_0}^{\hat\psi_0}\|^2$$

$$= \int \underbrace{\left|e^{-i\frac{k^2}{2}h}-1\right|^2}_{\xrightarrow{h\to 0} 0} |\hat\psi_0(k)|^2\,dk \xrightarrow{h\to 0} 0,$$

i.e., ψ is continuous for any $\hat\psi_0\in L^2$, i.e., $\psi\in L^2$ (by dominated convergence).

What about the map $P_f : \mathbb{R} \to \mathcal{B}(L^2)$?

$$\|P_f(t+h) - P_f(t)\|_{\mathcal{B}(L^2)} = \sup_{\substack{\varphi \in L^2 \\ \|\varphi\|=1}} \|P_f(t+h)\varphi - P_f(t)\varphi\|_{L^2}$$

$$= \sup_{\substack{\varphi \in L^2 \\ \|\varphi\|=1}} \left\| \left(e^{-i\frac{k^2}{2}(t+h)} - e^{-i\frac{k^2}{2}t}\right) \mathcal{F}\varphi \right\|_{L^2}$$

$$= \sup_{\substack{\tilde{\varphi} \in L^2 \\ \|\tilde{\varphi}\|=1}} \left\| \left(e^{-i\frac{k^2}{2}(t+h)} - e^{-i\frac{k^2}{2}t}\right) \tilde{\varphi} \right\|_{L^2}$$

Problem 4 Hw3:

$\|M_V\|_{\mathcal{B}(L^2)} = \|V\|_\infty$

$$= \sup_{k \in \mathbb{R}^d} \underbrace{\left| e^{-i\frac{k^2}{2}(t+h)} - e^{-i\frac{k^2}{2}t} \right|}_{= |e^{-i\frac{k^2}{2}h} - 1|}$$

$$= 2 \quad \text{for all } h \neq 0$$

So $P_f : \mathbb{R} \to \mathcal{B}(L^2)$ is not continuous.

We will come back to the discussion of P_f in the next section, but for now our upshot is that we need different notions of convergence.

<u>Definition 3.26:</u> Let $(A_n)_n$ be a sequence in $\mathcal{B}(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})$.

a) $(A_n)_n$ converges in norm (or "uniformly") to A if $\lim\limits_{n\to\infty} \|A_n - A\|_{\mathcal{B}(\mathcal{H})} = 0$.

Notation: $\lim\limits_{n\to\infty} A_n = A$, or $A_n \longrightarrow A$.

b) $(A_n)_n$ converges strongly (or "pointwise") to A if $\lim\limits_{n\to\infty} \|A_n\psi - A\psi\|_{\mathcal{H}} = 0 \ \forall \psi \in \mathcal{H}$.

Notation: $\operatorname{s-lim}\limits_{n\to\infty} A_n = A$, or $A_n \xrightarrow{s} A$.

c) $(A_n)_n$ converges weakly to A if $\lim_{n\to\infty} |\langle \varphi, (A_n - A)\psi\rangle| = 0 \quad \forall \varphi, \psi \in \mathcal{H}$.

Notation: $\text{w-}\lim_{n\to\infty} A_n = A$, or $A_n \xrightarrow{w} A$.

Note: • $|\langle \varphi, (A_n - A)\psi\rangle| \leq \|\varphi\| \, \|A_n\psi - A\psi\|_{\mathcal{H}} \leq \|\varphi\| \, \|\psi\| \, \|A_n - A\|_{\mathcal{B}(\mathcal{H})}$,

so norm convergence $\Rightarrow$ strong convergence $\Rightarrow$ weak convergence.

But the other way around is not true; come up with counterexamples in HW 6, Problem 1

• example above: $P_f(t)$ is strongly continuous (and thus weakly) but not norm continuous