Session 17
March 30, 2020

Last time we defined strongly continuous unitary one-parameter groups and their generators. This was motivated by our knowledge of the free Hamiltonian $H = -\frac{\Delta}{2}$ and the free propagator $P_f(t) = \mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F}$:
H with domain $\mathcal{D}(H) = H^2(\mathbb{R}^d)$ is generator of $P_f(t)$. Now let us give another example.

Example: Translation operator

- Let us consider $T(t): L^2(\mathbb{R}) \to L^2(\mathbb{R})$, where $(T(t)\psi)(x) := \psi(x-t)$.

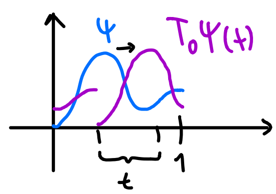
  We already introduced this operator on $\mathcal{S}$ as the pseudodifferential operator $e^{-it(-i\frac{d}{dx})}$. So we would guess that $D_0 = -i\frac{d}{dx}$ with domain $\mathcal{D}(D_0) = H^1(\mathbb{R})$ is the generator of the strongly continuous unitary one-parameter group $T(t)$. This is indeed so; proof in HW 7.

- How can we define translations on $L^2([0,1])$ as a unitary group? Unitarity means isometry (+surjective), so if we shift $\underbrace{L^2\text{-mass}}_{\text{some } \int |\psi|^2}$ out on one side of $[0,1]$ it needs to be put back on the other side.

So let us define, for $t \in [0,1)$ and some fixed phase factor $\theta \in [0, 2\pi)$,

$$T_\theta(t) = \begin{cases} \psi(x-t) & \text{if } x-t \in [0,1] \\ e^{i\theta}\psi(\underbrace{x-t}_{}+1) & \text{if } x-t < 0 \end{cases}$$

shifted out on the left; the +1 puts it back in



The phase factor θ is just an extra freedom we have.
The T_θ defined above is clearly unitary, and we define T_θ for all $t \in \mathbb{R}$ by the group property, given the def. for $t \in [0,1)$ (e.g. if $t, s \in [0,1)$, then T_θ is def. on $[0,2)$ by $T_\theta(t+s) = T_\theta(t) T_\theta(s)$).

Now, for $\theta \neq \theta'$, we have $T_\theta(t) \neq T_{\theta'}(t)$ at $t \neq 0$, i.e., as operators $T_\theta \neq T_{\theta'}$. So according to Proposition 3.33 iv) their generators are also different! But how can that be? T_θ still only shifts, so should not $-i\frac{d}{dx}$ be the generator? The answer is yes, but $-i\frac{d}{dx}$ can have different domains! Consider $D_\theta : \mathcal{D}(D_\theta) \to L^2([0,1])$, $\psi \mapsto -i\frac{d}{dx}\psi$, with domain

$$\mathcal{D}(D_\theta) = \{\psi \in \underbrace{H^1([0,1])}_{} : e^{i\theta}\underbrace{\psi(1)}_{} = \underbrace{\psi(0)}_{}\}.$$

$H^1([0,1]) := \{\psi \in L^2([0,1]) : \exists \varphi \in H^1(\mathbb{R})$ s.t. $\varphi|_{[0,1]} = \psi\}$

indeed ψ is defined pointwise because of the Sobolev lemma: $H^1(\mathbb{R}) \subset C(\mathbb{R})$.

Then we have $T_\theta(t)\mathcal{D}(D_\theta) = \mathcal{D}(D_\theta)$ (check this), and D_θ is the generator of T_θ.

Conclusion: We have to be very careful with choosing the right domain.

In practice we would often like to choose the domains "small", e.g., very nice/regular/smooth functions. But if the domain is too small, the operator might not be a generator: In the example above, $D_{min} := \{\psi \in H^1([0,1]) : \psi(0) = 0 = \psi(1)\}$ is not invariant under any T_θ, so it is not a generator. So then one has to enlarge the domain. But note that we cannot necessarily do that in a unique way: In the example above, $D_\theta \supset D_{min}$ $\forall \theta \in [0, 2\pi)$, so there are many possibilities. Also, one cannot enlarge the domain too much: In the example above $D_{max} := H^1([0,1])$ is again not a generator, since it is not invariant under any T_θ.

$\Rightarrow$ This will be the trade-off one has to be careful about in applications.

One more remark: Let us check the symmetry on the different domains (necessary condition for generators according to Proposition 3.33 iii)!)

For $\psi,\varphi \in H^1([0,1])$ we find: $\langle \psi, -i\frac{d}{dx}\varphi\rangle = \int_0^1 \overline{\psi(x)}\left(-i\frac{d}{dx}\varphi(x)\right)dx$

$$\overset{\text{integration by parts}}{=} -i\left(\overline{\psi(1)}\varphi(1) - \overline{\psi(0)}\varphi(0)\right) + \langle -i\tfrac{d}{dx}\psi, \varphi\rangle$$

We conclude:
- $-i\frac{d}{dx}$ not symmetric on $D_{max} = H^1([0,1])$ (boundary terms do not vanish), so $-i\frac{d}{dx}$ with domain D_{max} is not a generator
- on D_θ and D_{min}, $-i\frac{d}{dx}$ is symmetric (boundary terms vanish). But on D_{min} it is not a generator, so symmetry is a necessary but not sufficient condition.

In order to get a necessary and sufficient condition, we need to refine the concept of a symmetric operator. The right class of operators are self-adjoint operators (which in particular are symmetric).

(As a side note, in finite dimensional Hilbert spaces symm. and self-adjoint operators are the same. Recall from Linear Algebra that any self-adjoint map ($\bar{A}^T = A$ for the matrix) generates the unitary one-parameter group e^{-itA}. This can be easily seen by diagonalization.)

3.3 Self-adjoint Operators

Recall the general definition of the adjoint (here for normed spaces):

Definition 3.38: Let V and W be normed spaces and $A \in \mathcal{L}(V,W)$. Then the adjoint operator $A': W' \to V'$ (where V' and W' are the dual spaces of V and W) is defined by

$$A'(w')(v) = w'(Av) \quad \forall\, v \in V.$$

Note:
- For any normed space V, the dual space V' is a Banach space (even if V is not). This is so because elements of V' are continuous, i.e., bounded operators $V \to \mathbb{C}$, and $\mathbb{C}$ is complete (cf., Proposition 3.17).
- $A' \in \mathcal{L}(W', V')$ due to the definition
- With the Hahn-Banach theorem one can show that in fact $\|A'\|_{\mathcal{L}(W',V')} = \|A\|_{\mathcal{L}(V,W)}$.

Hilbert spaces are particularly nice because $\mathcal{H}'$ is isometrically isomorphic to $\mathcal{H}$. (We already noted that $L^p \cong (L^q)'$, $\frac{1}{p}+\frac{1}{q}=1$ in HW 3, so $L^2 \cong (L^2)'$.) So for $A \in \mathcal{L}(\mathcal{H})$, we would like to identify the operator $A' \in \mathcal{L}(\mathcal{H}')$ with an operator on $\mathcal{H}$. Let us first establish this connection; then we can introduce the notion of self-adjointness.

The key theorem is:

Theorem 3.39: The Riesz Representation Theorem

Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{H}'$. Then there is a unique $\psi_T \in \mathcal{H}$ s.t.

$$T(\varphi) = \langle \psi_T, \varphi \rangle_{\mathcal{H}} \quad \forall\, \varphi \in \mathcal{H}.$$

Proof: (Maybe you already saw this in Real Analysis?)

First, if $T(\varphi)=0 \ \forall \varphi \in \mathcal{H}$, then $T=0$ and $\psi_T=0$ is the unique vector in the theorem.
Otherwise, we want to show that T is the projection on the one-dimensional subspace spanned by some ψ_T.
So if we consider the kernel $M=\ker(T)$ $(:=\{\varphi\in\mathcal{H}: T(\varphi)=0\})$, a closed subspace of $\mathcal{H}$ (since T is continuous), we need to show that $M^\perp$ is one-dimensional. If $M=\mathcal{H}$, i.e., $\dim M^\perp=0$, then $\psi_T=0$, so let us assume $\dim M^\perp>0$.

But this follows directly from linearity: Let $\psi,\tilde\psi\in M^\perp\setminus\{0\}$. Then for $\alpha\in\mathbb{C}$,

$$T(\psi-\alpha\tilde\psi)=T(\psi)-\alpha T(\tilde\psi), \text{ so for } \alpha=\frac{T(\psi)}{T(\tilde\psi)}, \text{ we have } T(\psi-\alpha\tilde\psi)=0, \text{ i.e.,}$$

$$\psi-\alpha\tilde\psi\in M, \text{ so } \psi-\alpha\tilde\psi\in M\cap M^\perp=\{0\} \text{ and } \psi=\alpha\tilde\psi.$$

unique: $\frac{\langle\alpha\tilde\psi,\varphi\rangle}{\|\alpha\tilde\psi\|^2}\alpha\tilde\psi=\frac{\langle\tilde\psi,\varphi\rangle}{\|\tilde\psi\|^2}\tilde\psi \ \forall\alpha\in\mathbb{R}, \alpha\neq 0.$

Now we can uniquely decompose (with Theorem 3.15) any $\varphi=\varphi_M+\varphi_{M^\perp}=\varphi_M+\frac{\langle\tilde\psi,\varphi\rangle}{\|\tilde\psi\|^2}\tilde\psi$ for any $\tilde\psi\in M^\perp\setminus\{0\}$, and thus

$$T(\varphi)=T\left(\varphi_M+\frac{\langle\tilde\psi,\varphi\rangle}{\|\tilde\psi\|^2}\tilde\psi\right)\overset{T(\varphi_M)=0}{=}\frac{\langle\tilde\psi,\varphi\rangle}{\|\tilde\psi\|^2}T(\tilde\psi)=\left\langle\frac{\overline{T(\tilde\psi)}}{\|\tilde\psi\|^2}\tilde\psi,\varphi\right\rangle, \text{ i.e., } \psi_T=\frac{T(\tilde\psi)}{\|\tilde\psi\|^2}\tilde\psi. \quad \square$$