Session 20
April 20, 2020

We continue our discussion of self-adjointness for unbounded operators.

Today we discuss a few more properties and introduce some useful definitions. Those will lead us to our first set of criteria for self-adjointness.

First, recall the notion of a direct sum:

Definition 3.60: For two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, we define the direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}_1 \times \mathcal{H}_2$ and introduce the scalar product $\langle \varphi, \psi \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} = \langle \varphi_1, \psi_1 \rangle_{\mathcal{H}_1} + \langle \varphi_2, \psi_2 \rangle_{\mathcal{H}_2}$, where $\varphi = (\varphi_1, \varphi_2)$, $\psi = (\psi_1, \psi_2)$. (I.e., $\langle (\varphi_1, \varphi_2), (\psi_1, \psi_2) \rangle = \langle \varphi_1, \psi_1 \rangle + \langle \varphi_2, \psi_2 \rangle$).

Note: $\mathcal{H}_1 \oplus \mathcal{H}_2$ with this scalar product is of course a Hilbert space.

It turns out that the adjoint operator has, due to its definition via the scalar product, a nice property that gets us somewhat close to continuity (recall that unbounded operators are exactly the not continuous ones). Let us first define this property:

Definition 3.61:

a) The graph of a linear operator $(T, \mathcal{D}(T))$ is $\Gamma(T) = \{ (\varphi, T\varphi) \in \mathcal{H} \oplus \mathcal{H} : \varphi \in \mathcal{D}(T) \}$.

b) $\underbrace{(T, \mathcal{D}(T))}$ is called closed if $\Gamma(T)$ is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$.

often I will simply write T instead of $(T, \mathcal{D}(T))$

Put differently: T is closed if $\forall (\varphi_n)_n$ in $\mathcal{D}(T)$, $\varphi_n \to \varphi$ and $T\varphi_n \to \eta$ implies that $(\varphi, \eta) \in \Gamma(T)$, i.e., $\varphi \in \mathcal{D}(T)$ and $T\varphi = \eta$.

(Note: This is not continuity! For unbounded operators there are surely sequences $(\varphi_n)_n$ in $\mathcal{D}(T)$ such that $(T\varphi_n)_n$ does not converge.)

c) $(T, D(T))$ is called closeable if it has a closed extension. The smallest closed extension $\overline{T}$ is called closure of T. (Smallest in the sense of subsets and the domain. Note that the closure is unique, since we just need to add the limit points $(\varphi_n, T\varphi_n)$ to the graph.)

Note that $\Gamma(\overline{T}) = \overline{\Gamma(T)}$.

At this point, the property of being closed is a bit cumbersome to check. But a simple example would be:

↳ $(-\Delta, H^2(\mathbb{R}^d))$ is closed

↳ $(-\Delta, H^4(\mathbb{R}^d))$ is not closed, but it is closeable and $\overline{(-\Delta, H^4(\mathbb{R}^d))} = (-\Delta, H^2(\mathbb{R}^d))$.

The reason we introduced the notion of closedness is this:

<u>Theorem 3.63:</u> For densly defined $(T, D(T))$, the adjoint T^* is closed.

<u>Proof:</u> We need to show that $\Gamma(T^*)$ is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$. The idea is to write it as an orthogonal complement, because $\underbrace{\text{those are always closed}}_{\text{due to linearity and continuity of the scalar product}}$. Let's check:

$$
\begin{aligned}
(\psi, \eta) \in \Gamma(T^*) &\overset{\text{(def. of adjoint)}}{\iff} \langle \psi, T\varphi \rangle = \langle \eta, \varphi \rangle \quad \forall \varphi \in D(T) \\
&\iff -\langle \psi, T\varphi \rangle - \langle \eta, \varphi \rangle = 0 \quad \forall \varphi \in D(T) \\
&\iff \langle (\psi, \eta), (-T\varphi, \varphi) \rangle_{\mathcal{H} \oplus \mathcal{H}} = 0 \quad \forall \varphi \in D(T) \\
&\iff \langle (\psi, \eta), \phi \rangle_{\mathcal{H} \oplus \mathcal{H}} = 0 \quad \forall \phi \in W(\Gamma(T)), \text{ where}
\end{aligned}
$$

$W: \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H} : (\varphi_1, \varphi_2) \mapsto (-\varphi_2, \varphi_1)$, which is clearly unitary

Therefore, $\Gamma(T^*) = (W(\Gamma(T)))^\perp$, which is closed (since it is an orthogonal complement). □

Next, what if T is additionally symmetric? Then we have the following:

Proposition 3.64: A densly defined operator T is symmetric if and only if its adjoint is an extension, i.e., $T \subset T^*$.

Proof: "$\Rightarrow$" T symmetric $\Rightarrow \langle \psi, T\varphi \rangle = \langle T\psi, \varphi \rangle \ \forall \varphi, \psi \in D(T)$

$\Rightarrow \langle \psi, T\varphi \rangle = \langle \eta, \varphi \rangle \ \forall \varphi \in D(T)$ holds with $\eta = T\psi = T^*\psi$

$\Rightarrow D(T) \subset D(T^*)$

"$\Leftarrow$" for $\psi \in D(T) \subset D(T^*) \Rightarrow \forall \varphi \in D(T)$ we have $\langle \psi, T\varphi \rangle = \langle T^*\psi, \varphi \rangle = \langle T\psi, \varphi \rangle$,

so T symmetric. $\square$

Note:
- So together with Theorem 3.63, we find that symmetric operators are always closable.
- But for symmetric T, the closure $\overline{T}$ is not necessarily equal to T^* ($D(T^*)$ might be "too big"). In fact: T symmetric $\Rightarrow \overline{T}$ symmetric (see one of the next HW exercises), but T^* is usually not. (If T^* is also symmetric, then $T^* = \overline{T}$.)
- Try to connect these statements with the example of the translation operator on $[0,1]$.

Let us note a few more properties:

Proposition 3.67: For densly defined T and $T \subset S$ we have $S^* \subset T^*$.

So extending a densly defined operator makes the adjoint "smaller".

Proof: With W as in the proof of Theorem 3.63, we have $\Gamma(S^*) = (W(\Gamma(S)))^\perp$.

But now $T \subset S$, i.e., $\Gamma(T) \subset \Gamma(S)$, so also $W(\Gamma(T)) \subset W(\Gamma(S))$, so

$$\Gamma(S^*) = (W(\Gamma(S)))^\perp \subset (W(\Gamma(T)))^\perp = \Gamma(T^*). \qquad \square$$

We already noted that adjoints of densly defined operators are not necessarily densly defined themselves. But:

Proposition 3.68: If a densly defined (linear) operator T is closable, then its adjoint T^* is densly defined.

Proof: Using what we already established so far, let us show denseness of $D(T^*)$ by showing that $(D(T^*))^\perp = 0$. So let us choose some $\eta \in (D(T^*))^\perp$.

Then by definition $\langle \eta, \varphi \rangle = 0 \ \forall \varphi \in D(T^*)$, so in particular $\langle (\eta, 0), (\varphi, T^*\varphi) \rangle = 0$ $\forall \varphi \in D(T^*)$, so $(\eta, 0) \in (\underbrace{\Gamma(T^*)}_{=(W(\Gamma(T)))^\perp \text{ from proof of Thm. 3.63}})^\perp = (W(\Gamma(T)))^{\perp\perp} = \overline{W(\Gamma(T))}$.

But $W(\Gamma(T)) = \{(-T\varphi, \varphi) : \varphi \in D(T)\}$, so there must be a sequence in $W(\Gamma(T))$ that converges to $(\eta, 0)$, i.e., $(\varphi_n)_n$ in $D(T)$ with $\varphi_n \to 0$ and $-T\varphi_n \to \eta$.

But now T is closable, so $\overline{T}\, 0 = \eta$, which equals 0 since $\overline{T}$ is linear. So $\eta = 0$. $\square$

A few more properties:

Proposition 3.69: If T is densly defined and closable, then

a) $T^{**} = \overline{T}$

b) $(\overline{T})^* = T^* = T^{***}$.

Let us skip the proof, it is similar to the previous ones.

Finally, what does this imply for symmetric operators?

$\boxed{\begin{array}{l}\text{Corollary 3.70: Let } T \text{ be densly defined and symmetric. Then } T^* \text{ is closable and densly} \\ \text{defined, and } \quad a)\ T \subset \overline{T} = T^{**} \subset T^* = T^{***}, \text{ in particular } T^{**} \text{ is symmetric.} \\ \qquad\qquad\qquad\quad b)\ (\overline{T})^* = T^*\end{array}}$

This follows immediately from the properties we have discussed so far.