Session 22
April 27, 2020

Today, we get a bit more concrete with conditions for self-adjointness and essential self-adjointness.

Nice quantities to look at are the kernel of $H^* + z$ and the image of $H + z$, for $z \in \mathbb{C}$.

Let us prove one auxiliary lemma, and then state our conditions.

Lemma 3.79: Let $(T, \mathcal{D}(T))$ be a densly defined linear operator. Then

a) $\ker(T^* \pm z) = \operatorname{im}(T \pm \bar{z})^\perp \quad \forall z \in \mathbb{C}$; in particular:

$$\ker(T^* \pm z) = \{0\} \iff \overline{\operatorname{im}(T \pm \bar{z})} = \mathcal{H}.$$

b) If T is closed and symmetric, then $\operatorname{im}(T \pm i)$ is closed.

Proof: a) $\psi \in \operatorname{im}(T \pm \bar{z})^\perp \iff \langle \psi, (T \pm \bar{z})\varphi \rangle = 0 \quad \forall \varphi \in \mathcal{D}(T)$

$$\iff \psi \in \mathcal{D}(T^*) \text{ and } (T^* \pm z)\psi = 0$$

$$\iff \psi \in \ker(T^* \pm z)$$

b) If T symmetric, then $\langle \psi, T\psi \rangle = \langle T\psi, \psi \rangle = \overline{\langle \psi, T\psi \rangle}$, so $\langle \psi, T\psi \rangle \in \mathbb{R} \quad \forall \psi \in \mathcal{D}(T)$.

Then $\|(T \pm i)\psi\|^2 = \langle (T \pm i)\psi, (T \pm i)\psi \rangle = \|T\psi\|^2 + \|\psi\|^2 \underbrace{\mp 2 \operatorname{Re} i \underbrace{\langle \psi, T\psi \rangle}_{\in \mathbb{R}}}_{= 0}$

$$= \|T\psi\|^2 + \|\psi\|^2 \geq \|\psi\|^2.$$

So $T \pm i$ is injective, so $(T \pm i)^{-1} : \operatorname{im}(T \pm i) \to \mathcal{D}(T)$ exists and is bounded.

Now let's check closedness of $\mathrm{im}(T \pm i)$. Let $(\psi_n)_n$ be a sequence in $\mathrm{im}(T \pm i)$, let $\psi_n \to \psi$.

Then $\varphi_n := (T \pm i)^{-1} \psi_n$ is a Cauchy sequence and $\varphi_n \to \varphi \in \mathcal{H}$. But T is closed, so also $\Gamma(T \pm i)$ is closed, so $(\psi_n, \varphi_n)_n$ as a sequence in $\Gamma(T \pm i)$ converges to $(\varphi, \psi) = (\varphi, (T \pm i)\varphi)$, so $\psi \in \mathrm{im}(T \pm i)$.

To summarize: Every sequence in $\mathrm{im}(T \pm i)$ converges in $\mathrm{im}(T \pm i)$, so it is closed. $\square$

Now the criteria:

Theorem 3.78: Let $(H, D(H))$ be a densly defined symmetric linear operator. Then the following statements are equivalent:

(i) H is self-adjoint.

(ii) H is closed and $\ker(H^* \pm i) = \{0\}$.

(iii) $\mathrm{im}(H \pm i) = \mathcal{H}$.

Proof:

(i) $\Rightarrow$ (ii) H self-adjoint $\Rightarrow H = H^* \Rightarrow H$ closed.

If $\varphi_\pm \in \ker(H^* \pm i)$ it means $(H^* \pm i)\varphi_\pm = 0$, i.e., $H\varphi_\pm = \mp i \varphi_\pm$, which can't be since eigenvalues of symmetric operators are real. So $\varphi_\pm = 0$.

(ii) $\Rightarrow$ (iii) This is Lemma 3.79 b)

(iii) $\Rightarrow$ (i) H symmetric, so $H \subset H^*$ (Proposition 3.64). So we still need to show that also $H^* \subset H$, i.e., $D(H^*) \subset D(H)$. So let's choose $\psi \in D(H^*)$.

By (iii) $\exists\, \varphi \in D(H)$ s.t. $\mathcal{H} \ni (H^* - i)\psi = (H - i)\varphi$.

But $H \subset H^*$, so also $(H^*-i)\psi = (H^*-i)\varphi$, i.e., $\varphi - \psi \in \ker(H^*-i)$.
So with Lemma 3.79 a) we have $\varphi - \psi = 0$, so $\psi = \varphi \in D(H)$. $\square$

Similar criteria hold for essential self-adjointness:

Corollary 3.80: Let $(H, D(H))$ be a densly defined symmetric linear operator. Then the following statements are equivalent:

(i) H is essentially self-adjoint.
(ii) $\ker(H^* \pm i) = \{0\}$.
(iii) $\overline{\operatorname{im}(H \pm i)} = \mathcal{H}$.

Proof: (ii) $\Leftrightarrow$ (iii) was Lemma 3.79 a).

Also: H essentially self-adjoint $\Leftrightarrow$ $\overline{H} \overset{\text{Proposition 3.69}}{=} H^{**}$ is self-adjoint

$\overset{\text{Theorem 3.78}}{\Leftrightarrow}$ H^{**} closed and $\ker(H^{***} \pm i) = \{0\}$

$\overset{\text{Proposition 3.69}}{\Leftrightarrow}$ $\ker(H^* \pm i) = \{0\}$,

so (i) $\Leftrightarrow$ (ii). $\square$

Examples:

- $D_{min} = -i \frac{d}{dx}$ with $D(D_{min}) = \{ \psi \in H^1([0,1]) : \psi(1) = 0 = \psi(0) \}$.

  Let us check $\ker(D^*_{min} \pm i) \ni \varphi_\pm$.

$(D_{min}^* \pm i)\varphi_\pm = 0$, i.e., $-i\frac{d}{dx}\varphi_\pm = \mp i\varphi_\pm$, i.e., $\frac{d}{dx}\varphi_\pm = \pm\varphi_\pm$, which has solutions $\varphi_\pm = e^{\pm x} \in \mathcal{D}(D_{min}^*) = H^1([0,1])$. So there are non-zero elements in the kernel ($\dim \ker(D_{min}^* \pm i) = 1$ in fact), so again we see that D_{min} is not essentially self-adjoint.

- $H_0 = -\Delta$ with $\mathcal{D}(H_0) = C_0^\infty(\mathbb{R}^d)$. Then $\mathcal{D}(H_0^*) = H^2(\mathbb{R}^d)$. Let us check $\ker(H_0^* \pm i)$ again: $(H_0^* \pm i)\varphi_\pm = 0$, i.e., $H_0^*\varphi_\pm = -\Delta\varphi_\pm = \mp i\varphi_\pm$. But now $\varphi_\pm(x) = e^{xk}$ with $k^2 = \pm i$, so $\varphi_\pm$ are <u>not</u> in $H^2(\mathbb{R}^d)$. So $\ker(H_0^* \pm i) = \{0\}$, i.e., H_0 is essentially self-adjoint.

  We already saw that its closure is $(-\Delta, H^2(\mathbb{R}^d)) = (H_0^*, H^2(\mathbb{R}^d))$.

Next time we will discuss how to deal with multiple possible self-adjoint extensions, and then we will prove the important Kato-Rellich theorem.

((Later, we will also give an example of a symmetric operator that does not have any self-adjoint extensions.)