Session 23
April 29, 2020

Last time we discussed criteria for self-adjointness and essential self-adjointness of a densly defined symmetric linear operator $(H, D(H))$.
We found that

(i) H is self-adjoint. $\iff$ (ii) H is closed and $\ker(H^* \pm i) = \{0\}$.

$\iff$ (iii) $\mathrm{im}(H \pm i) = \mathcal{H}$.

(i) H is essentially self-adjoint. $\iff$ (ii) $\ker(H^* \pm i) = \{0\}$.

$\iff$ (iii) $\overline{\mathrm{im}(H \pm i)} = \mathcal{H}$.

Today, we will deal with the possibility of multiple self-adjoint extensions (as we had for $D_{min} = -i\frac{d}{dx}$). For that, we need to look a bit closer at $\ker(H^* \pm i)$ and $\mathrm{im}(H \pm i)$.
Note that $\underbrace{\ker(H^* \mp i) = \mathrm{im}(H \pm i)^\perp}$ by definition of H^*. The dimensions of these spaces have names:

$\eta \in \ker(H^* \mp i) \iff (H^* \mp i)\eta = 0 \iff \langle (H^* \mp i)\eta, \psi\rangle = 0 \ \forall \psi \in D(H) \iff \langle \eta, (H \pm i)\psi\rangle = 0 \ \forall \psi \in D(H)$

Definition 3.82: Let $(H, D(H))$ be a densly defined symmetric linear operator. Then

- $L^\pm := \ker(H^* \mp i) = \mathrm{im}(H \pm i)^\perp$ are called deficiency spaces of H,
- $N^\pm := \dim L^\pm$ are called deficiency indices of H.

So essential self-adjointness $\iff N^+ = N^- = 0 \quad (\mathcal{H} = \overline{\mathrm{im}(H \pm i)} \oplus L^\pm)$.

What we want to show next is that existence of self-adjoint extensions $\iff N^+ = N^-$
(but not necessarily $= 0$)

These can be nicely studied with the following map:

Definition 3.84: Let $(H, \mathcal{D}(H))$ be a densly defined symmetric linear operator. Then we define its Cayley transform $U: \mathrm{im}(H+i) \to \mathrm{im}(H-i)$ as $U := (H-i)(H+i)^{-1}$.

Note: • U is clearly surjective

• U is also isometric, since

$$\|U\psi\| = \|U\underbrace{(H+i)\varphi}_{\substack{\text{by def. } \psi=(H+i)\varphi \\ \text{for some } \varphi}}\| = \|(H-i)\varphi\| = \sqrt{\|H\varphi\|^2 + \|\varphi\|^2} = \|(H+i)\varphi\| = \|\psi\|$$

Now, let us skip the details, and just give a summary of main result concerning the Cayley transform. This is:

Lemma 3.86: There is a one-to-one correspondence between the symmetric extensions of H and isometric extensions of its Cayley transform U.

Then, the following also holds:

Lemma 3.90: H self-adjoint $\iff$ Cayley transform U is unitary on $\mathcal{H}$.

These two lemmas imply:

Theorem 3.89: H has self-adjoint extensions $\iff N^+ = N^-$.

In this case the extensions are parametrized by $u \in U(N^+)$ (the unitary group, i.e., unitary $n \times n$ matrices).

We skipped the proofs, but let us look at examples:

- $D_{min} = -i\frac{d}{dx}$ on $\mathcal{H} = L^2([0,1])$ with $D(D_{min}) = C_0^\infty((0,1))$. We already found before that $\varphi_\pm = e^{\mp x} \in D(D_{min}^*) = H^1([0,1])$, so $L^\pm = \operatorname{span}\{\varphi_\pm\}$, i.e., $N^+ = N^- = 1$. So D_{min} has self-adjoint extensions parametrized by $U(1)$, i.e., multiplication with $e^{i\alpha}$.

- Translations on the half-line. Let $D_{[0,\infty)} = -i\frac{d}{dx}$ on $\mathcal{H} = L^2([0,\infty))$ with domain $D(D_{[0,\infty)}) = C_0^\infty((0,\infty))$. Integration by parts shows that $D_{[0,\infty)}$ is indeed symmetric, and $D(D_{[0,\infty)}^*) = H^1([0,\infty))$. Now consider $L^\pm = \ker(D^*_{[0,\infty)} \mp i)$, i.e., we need to find solutions to
$$D^*_{[0,\infty)}\varphi_\pm = -i\frac{d}{dx}\varphi_\pm = \pm i\varphi_\pm .$$
The only solutions are $\varphi_\pm = e^{\mp x}$. Now $\varphi_+ \in D(D^*_{[0,\infty)})$ since it goes to 0 at ∞. Thus $L^+ = \operatorname{span}\{e^{-x}\}$ and $N^+ = 1$. But φ_- is not even in $L^2([0,\infty))$, so $L^- = \{0\}$ and $N^- = 0$.

  Therefore, $D_{[0,\infty)}$ has no self-adjoint extensions. This makes sense because it is clearly impossible to define a unitary group of translations on $L^2([0,\infty))$.

  But note that $-\Delta$ on $L^2([0,\infty))$ can be defined as a self-adjoint operator, so the Schrödinger equation on the half-line makes sense, as we will see shortly.

Finally, let us mention a very useful criterion to check whether $N^+ = N^-$. For this, recall the following:

$$\boxed{\text{\underline{Definition 3.93:} An anti-linear map } C: \mathcal{H} \to \mathcal{H} \text{ (meaning that } C(\alpha\psi + \varphi) = \bar{\alpha}C(\psi) + C(\varphi)) \text{ is called a conjugation if } \|C\psi\| = \|\psi\|\ \forall \psi \in \mathcal{H} \text{ and } C^2 = \mathrm{id}_{\mathcal{H}}.}$$

Examples: • Of course, complex conjugation $\psi(x) \mapsto \overline{\psi(x)}$ is a conjugation.

• For Dirac spinors $\psi \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ that come up in relativistic quantum physics, $\psi(x) \mapsto \gamma\,\overline{\psi(x)}$ for any 4×4 matrix γ with $\gamma\overline{\gamma} = \mathrm{id}$, is a conjugation.

Then it follows relatively straightforwardly that the following holds:

Theorem 3.95: von Neumann-theorem

Let $(H, \mathcal{D}(H))$ be a densly defined symmetric linear operator. If there is a conjugation C with $C\mathcal{D}(H) \subset \mathcal{D}(H)$ and $CH\varphi = HC\varphi \;\forall \varphi \in \mathcal{D}(H)$, then $N^+ = N^-$ (i.e., H has self-adjoint extensions).

Examples: • $D_{[0,\infty)} = -i\frac{d}{dx}$ on $C_0^\infty([0,\infty))$ does not commute with complex conjugation, and in fact not with any conjugation.

• But $-\Delta$ on $C_0^\infty([0,\infty))$ does commute with complex conjugation, so indeed it has self-adjoint extensions.

The last technical tool that we need is the resolvent. This is a very important operator especially in spectral theory, but it will also be useful for us here.

Recall that for $n\times n$ matrices A we defined eigenvalues λ and eigenvectors $\vec{x} \neq 0$ as solutions to $A\vec{x} = \lambda\vec{x}$, i.e., $(A - \underbrace{\lambda}_{\text{meaning } \lambda\,\mathrm{id}})\,\vec{x} = 0$. These exist if $(A-\lambda)$ is not invertible.

This we can generalize to unbounded operators.

Definition 3.97: Let $(T, D(T))$ be a linear operator on $\mathcal{H}$. Then we define the resolvent set $\rho(T) := \{ z \in \mathbb{C} : (T-z) : D(T) \to \mathcal{H} \text{ is bijective with continuous inverse} \}$.

For $z \in \rho(T)$ we call $R_z(T) := (T-z)^{-1} \in \mathcal{L}(\mathcal{H})$ the resolvent of T at z.

The spectrum of T is $\sigma(T) = \mathbb{C} \setminus \rho(T)$ (the complement of the resolvent set).

We differentiate further:

a) $\sigma_p(T) := \{ z \in \mathbb{C} : T-z \text{ is not injective} \}$ is called the point spectrum, and any $z \in \sigma_p(T)$ is called eigenvalue of T.

b) $\sigma_c(T) := \{ z \in \mathbb{C} : T-z \text{ is injective, but not surjective, but has dense image} \}$ is called the continuous spectrum.

c) $\sigma_r(T) := \{ z \in \mathbb{C} : T-z \text{ is injective, but not surjective, and the image is not dense} \}$ is called residual spectrum.

Before we discuss examples, let us note the following:

Theorem 3.98:

- T closed $\Rightarrow \rho(T) = \{ z \in \mathbb{C} : (T-z) : D(T) \to \mathcal{H} \text{ is bijective} \}$ and for $z \in \rho(T)$ we have that $(T-z)^{-1} : \mathcal{H} \to (D(T), \|\cdot\|_{D(T)})$ is bounded.
- T not closed $\Rightarrow \rho(T) = \emptyset$.

Proof: If T is closed, $(D(T), \|\cdot\|_{D(T)})$ is complete, i.e., a Hilbert space. And by definition, $T : (D(T), \|\cdot\|_{D(T)}) \to \mathcal{H}$ is bounded, in fact by 1 in norm:
$\|T\psi\|_{\mathcal{H}}^2 \leq \|T\psi\|_{\mathcal{H}}^2 + \|\psi\|_{\mathcal{H}}^2 =: \|\psi\|_{D(T)}^2$.

Now one can use the open mapping theorem: Let $A: X \to Y$ be a continuous linear operator, where X, Y are Banach spaces. If A is surjective, then A is an open map (i.e., it maps open sets into open sets).

So here, if $T - z : (D(T), \|\cdot\|_{D(T)}) \to \mathcal{H}$ is surjective, it is also open. So if $T - z$ is bijective, then $(T-z)^{-1} : \mathcal{H} \to (D(T), \|\cdot\|_{D(T)})$ is continuous (preimages of open sets are open).

For the second statement one can show $\rho(T) \neq \emptyset \Rightarrow T$ closed. If $\rho(T) \neq \emptyset$, then there is a $z \in \mathbb{C}$ s.t. the resolvent $R_z(T) : \mathcal{H} \to D(T)$ exists. Using the definition of closedness one can then infer that T is indeed closed; let us skip the details. $\square$

So for closed T we do not need the requirement of continuous inverse in the definition of $\rho(T)$, and thus $\sigma(T) = \underbrace{\sigma_p(T) \,\dot\cup\, \sigma_c(T) \,\dot\cup\, \sigma_r(T)}_{\text{they are disjoint}}$.

And for $\dim \mathcal{H} < \infty$, $T - z$ injective $\Leftrightarrow T - z$ surjective, so $\sigma(T) = \sigma_p(T)$, the eigenvalues.