Session 24
May 4, 2020

As example, let us consider the "position operator" $\hat{x}: \psi(x) \mapsto x\,\psi(x)$

with domain $\mathcal{D}(\hat{x}) = \{\psi \in L^2(\mathbb{R}) : x\,\psi(x) \in L^2(\mathbb{R})\}$.

When is the resolvent $(\hat{x}-z)^{-1}$ bounded? Need $z \in \mathbb{C}\setminus\mathbb{R}$, so $\sigma(\hat{x}) = \mathbb{R}$.

Now for $\lambda \in \mathbb{R}$, $\hat{x}-\lambda$ has dense image: we could, for $\psi \in L^2(\mathbb{R})$ just use

$$\varphi_n(x) := \mathbb{1}_{\mathbb{R}\setminus[\lambda-\frac{1}{n}, \lambda+\frac{1}{n}]}(x)\,\frac{\psi(x)}{x-\lambda}\text{, i.e., } (x-\lambda)\,\varphi_n(x) \longrightarrow \psi(x) \in L^2(\mathbb{R}).$$

So $\hat{x}$ has only continuous spectrum: $\sigma(\hat{x}) = \sigma_c(\hat{x}) = \mathbb{R}$.

Note that the same holds true for the "momentum operator" $\hat{p} = -i\frac{d}{dx}$, since $\hat{p} = \mathcal{F}\hat{x}\mathcal{F}^{-1}$,

and $T-z$ bijective $\Leftrightarrow$ $U(T-z)U^{-1}$ bijective for $U$ unitary.

Next, let us establish more properties of the resolvent. After that, will put it to use for the question of self-adjointness (especially Kato-Rellich).

<u>Lemma 3.100:</u> Let $(T, \mathcal{D}(T))$ be a densly defined linear operator. Then

a) $\rho(T)$ is open, i.e., $\sigma(T)$ is closed

b) The map $\rho(T) \to \mathcal{B}(\mathcal{H})$, $z \mapsto R_z(T)$ is analytic

(i.e., it can be written as $R_z(T) = \sum_{n=0}^{\infty} (z-z_0)^n \underbrace{R_{z_0}^{n+1}}_{\in \mathcal{B}(\mathcal{H})}$ for some $z_0 \in \rho(T)$ with non-zero radius of convergence)

c) For $T \in \mathcal{B}(\mathcal{H})$, we have $|z| \leq \|T\|$ $\forall z \in \sigma(T)$ and thus $\sigma(T)$ is compact.

d) The first resolvent identity holds: for $z, w \in \varrho(T)$, we have

$$R_z(T) - R_w(T) = (z-w) R_w(T) R_z(T).$$

Thus, in particular $R_w(T) R_z(T) = R_z(T) R_w(T)$, i.e., the resolvents commute.

<u>Proof:</u> The proof uses the Neumann series from Homework 8, Problem 3:

(*): Let $T \in \mathcal{L}(X)$ with $\|T\| < 1$ and $X$ a Banach space. Then $1 - T$ is continuously invertible and

$$(1-T)^{-1} = \sum_{n=0}^{\infty} T^n \quad \text{and} \quad \|(1-T)^{-1}\| \leq (1 - \|T\|)^{-1}.$$

a) Let $z_0 \in \varrho(T)$ and choose $z$ s.t. $|z - z_0| < \|R_{z_0}(T)\|^{-1}$. Then

$T - z = T - z_0 + (z_0 - z) = (T - z_0)\Big(1 - (z - z_0) R_{z_0}(T)\Big)$. Since $\|(z-z_0) R_{z_0}(T)\| < 1$ by assumption, (*) tells us that $1 - (z - z_0) R_{z_0}(T)$ is continuously invertible, and thus also $T - z$. So also $z \in \varrho(T)$.

b) This again follows from (*): With the same notation as in a), we have

$$R_z(T) = \big(1 - (z-z_0) R_{z_0}(T)\big)^{-1} R_{z_0}(T) = \sum_{n=0}^{\infty} (z - z_0)^n R_{z_0}(T)^{n+1}.$$

c) If $|z| > \|T\|$, then $\|\frac{T}{z}\| < 1$, i.e., $1 - \frac{T}{z}$ is invertible, and thus also $T - z$, so $z \in \varrho(T)$. So $z \notin \varrho(T) \Rightarrow |z| \leq \|T\|$.

d) We compute: $R_z(T)-R_w(T) = R_z(T)\underbrace{(T-w)R_w(T)}_{=\mathrm{id}} - R_z(T)\underbrace{(T-z)R_w(T)}_{=\mathrm{id}}$

$$= (z-w)\, R_z(T) R_w(T). \qquad \square$$

Self-adjoint operators have real spectrum, and there is a simple bound for the resolvent:

**Lemma 3.102:** Let $H$ be self-adjoint. Then $\sigma(H) \subset \mathbb{R}$ and for $z \in \mathbb{C}\setminus\mathbb{R}$ we have

$$\|(H-z)^{-1}\| \leq \frac{1}{|\operatorname{Im}(z)|}.$$

**Proof:** Let us write $z = \lambda + i\mu$ with $\lambda, \mu \in \mathbb{R}$, $\mu \neq 0$. Then $\frac{H-\lambda}{\mu}$ is still self-adjoint (with the same domain) and thus with Theorem 3.78:

$$\{0\} = \ker\left(\frac{H-\lambda}{\mu} - i\right) = \ker(\overbrace{H-\lambda-i\mu}^{H-z}), \text{ so } H-z \text{ is injective, and}$$

$$\mathcal{H} = \operatorname{im}\left(\frac{H-\lambda}{\mu} - i\right) = \operatorname{im}(\overbrace{H-\lambda-i\mu}^{H-z}), \text{ so } H-z \text{ is surjective.}$$

So $H-z: D(H) \to \mathcal{H}$ is bijective.

Furthermore: $\|(\overbrace{H-\lambda-i\mu}^{H-z})\psi\|^2 = \|(H-\lambda)\psi\|^2 + \|\mu\psi\|^2 \geq \mu^2\|\psi\|^2$, so

with $\psi = (H-z)^{-1}\varphi$ we find that $\|(H-z)^{-1}\| \leq \frac{1}{|\mu|}$. $\square$

Note: One can actually show something stronger, namely that $\|(H-z)^{-1}\| = \frac{1}{\operatorname{dist}(z, \sigma(H))}$.

Finally, let us state Kato-Rellich. The idea is that we often consider operators $H = H_0 + V$, where we know that $H_0$ is self-adjoint with domain $D(H_0)$ and we want to know whether also $H$ is self-adjoint. Often the $V$'s are such that they do not disturb $D(H_0)$ too much, i.e., they do not introduce new boundary points. For example, $H_0 = -\Delta$ and $V = \frac{1}{|x|}$ for the hydrogen atom.

<u>Definition:</u> Let $A, B$ be densly defined linear operators with $D(A) \subset D(B)$ and such that there are $a, b \geq 0$ s.t.

$\|B\varphi\| \leq a\|A\varphi\| + b\|\varphi\| \quad \forall \varphi \in D(A)$. Then $B$ is called relatively bounded by $A$ (or $A$-bounded), and the infimum over all permissible $a$ is called the relative bound.

If the relative bound $= 0$, $B$ is called infinitesimally $A$-bounded.

Note: In Homework 10, Problem 4, we show that one can equivalently ask for

$\|B\varphi\|^2 \leq \tilde{a}\|A\varphi\|^2 + \tilde{b}\|\varphi\|^2$; this leads to the same relative bound.

Finally:

<u>Theorem (Kato-Rellich):</u> Let $A$ be self-adjoint, $B$ symmetric and $A$-bounded with relative bound $a < 1$. Then $A+B$ is self-adjoint on $D(A+B) = D(A)$ and essentially self-adjoint on every core of $A$.

Proof: Next time.