Jacobs University September 15, 2021
Fall 2021

Foundations of Mathematical Physics

Homework 1

Due on September 22, 2021

Problem 1 [4 points]: Integrals with Parameter
Prove the following lemma that was stated in class. Let $\Gamma \subset \mathbb{R}$ be an open interval and $f : \mathbb{R}^d \times \Gamma \to \mathbb{C}$ such that $f(x, \gamma) \in L^1(\mathbb{R}^d)$ for all fixed $\gamma \in \Gamma$. Let $I(\gamma) = \int_{\mathbb{R}^d} f(x, \gamma)dx$.

(a) If the map $\gamma \mapsto f(x, \gamma)$ is continuous for almost all $x \in \mathbb{R}^d$ and if there is a $g \in L^1(\mathbb{R}^d)$ with $\sup_{\gamma\in\Gamma} |f(x, \gamma)| \leq g(x)$ for almost all $x \in \mathbb{R}^d$, then $I(\gamma)$ is continuous.

(b) If the map $\gamma \mapsto f(x, \gamma)$ is continuously differentiable for all $x \in \mathbb{R}^d$ and if there is a $g \in L^1(\mathbb{R}^d)$ with $\sup_{\gamma\in\Gamma} |\partial_\gamma f(x, \gamma)| \leq g(x)$ for almost all $x \in \mathbb{R}^d$, then $I(\gamma)$ is continuously differentiable and

$$\frac{dI(\gamma)}{d\gamma} = \frac{d}{d\gamma}\int_{\mathbb{R}^d} f(x, \gamma)dx = \int_{\mathbb{R}^d} \frac{\partial}{\partial\gamma} f(x, \gamma)dx.$$

Problem 2 [4 points]: Integration by Parts in L^1
Let $f \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$ with $f' \in L^1(\mathbb{R})$.

(a) Show that $f \in C_\infty(\mathbb{R})$.

(b) Show integration by parts for $g \in L^\infty(\mathbb{R}) \cap C^1(\mathbb{R})$ with $g' \in L^\infty(\mathbb{R})$, i.e.,

$$\int_{\mathbb{R}} g(x)f'(x)dx = -\int_{\mathbb{R}} g'(x)f(x)dx.$$

(c) Show that this implies that $\widehat{f'}(k) = ik\widehat{f}(k)$.

Problem 3 [4 points]: Smoothness and Decay of the Fourier Transform
Let $\ell \in \mathbb{N}$ and $f \in L^1(\mathbb{R})$. State (for each case separately) sufficient and preferably weak conditions for f which imply that

(a) $\widehat{f} \in C^\ell(\mathbb{R})$,

(b) $\sup_{k\in\mathbb{R}} \left| |k|^\ell \widehat{f}(k) \right| < \infty$,

(c) $\widehat{f} \in L^1(\mathbb{R})$.

Problem 4 [4 points]: Fourier Transform of a Gaussian
Let $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$ and $f : \mathbb{R} \to \mathbb{C}$ defined by $f(x) = e^{-ax^2/2}$. Show that $f \in \mathcal{S}(\mathbb{R})$ and

$$\widehat{f}(k) = a^{-1/2} e^{-k^2/(2a)}.$$

Problem 5 [4 points]: Bosons and Fermions
We consider two non-interacting particles in one dimension confined to the box $[-\frac{1}{2}, \frac{1}{2}]$, with periodic boundary conditions.

(a) Consider just one of the particles, and find the ground state $\varphi_0(x)$ and first excited state $\varphi_1(x)$. (The time-independent Schrödinger equation is $E\varphi = -\frac{1}{2}\Delta\varphi$.)

(b) For fermions, the 2-particle ground state is

$$\psi_{\text{fermion}}(x_1, x_2) = \frac{1}{\sqrt{2}}\Big(\varphi_0(x_1)\varphi_1(x_2) - \varphi_1(x_1)\varphi_0(x_2)\Big).$$

Consider also the bosonic state with the same energy (which here is the first 2-particle excited state)

$$\psi_{\text{boson}}(x_1, x_2) = \frac{1}{\sqrt{2}}\Big(\varphi_0(x_1)\varphi_1(x_2) + \varphi_1(x_1)\varphi_0(x_2)\Big).$$

For each of these two states, what is the probability that both particles are on the same side of the box, i.e., that *both* particles are in $[-\frac{1}{2}, 0]$ or in $[0, \frac{1}{2}]$?

(c) Compute the probability above also for the bosonic 2-particle ground state.