

# Foundations of Mathematical Physics

## Homework 2

Due on September 29, 2021

### Problem 1 [6 points]: Bijectivity of the Fourier Transform

Finish the proof that was sketched in class, that the Fourier transform  $\mathcal{F} : \mathcal{S} \rightarrow \mathcal{S}$  is a continuous bijection with continuous inverse  $\mathcal{F}^{-1}$  (with  $\mathcal{F}^{-1}$  as defined in class).

### Problem 2 [2 points]: Plancherel on $\mathcal{S}$

Prove that for  $f, g \in \mathcal{S}(\mathbb{R}^d)$ ,

$$\int_{\mathbb{R}^d} \widehat{f}(x)g(x)dx = \int_{\mathbb{R}^d} f(x)\widehat{g}(x)dx,$$

and in particular,

$$\int_{\mathbb{R}^d} |\widehat{f}(k)|^2 dk = \int_{\mathbb{R}^d} |f(x)|^2 dx.$$

### Problem 3 [6 points]: Dilations

Let  $p \in [1, \infty)$ ,  $\sigma > 0$  and  $D_\sigma^p : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ ,  $f(x) \mapsto (D_\sigma^p f)(x) = \sigma^{-d/p} f(x/\sigma)$  the  $L^p$  dilation with  $\sigma$ .

- (a) Show that  $D_\sigma^p : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$  is continuous and  $\|D_\sigma^p f\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}$ .
- (b) Compute  $\mathcal{F}D_\sigma^2 f$  and interpret the result.

### Problem 4 [6 points]: Convolution

Let  $f \in C(\mathbb{R}^d)$  be bounded and let  $\varphi \in \mathcal{S}(\mathbb{R}^d)$  with  $\varphi \geq 0$  and  $\|\varphi\|_{L^1(\mathbb{R}^d)} = 1$ . Define  $f_\sigma := f * (D_\sigma^1 \varphi)$ , where the  $D_\sigma^1$  are the dilations from Problem 3.

- (a) Show that  $f_\sigma \in C^\infty(\mathbb{R}^d)$ .
- (b) Show that  $f_\sigma(x) \rightarrow f(x)$  as  $\sigma \rightarrow 0$  for all  $x \in \mathbb{R}^d$ .
- (c) Show that the convergence in b) is uniform on each compact interval. You may additionally assume  $\varphi \in C_c^\infty(\mathbb{R}^d)$ .