Jacobs University October 6, 2021
Fall 2021

Foundations of Mathematical Physics

Homework 4

Due on October 13, 2021

Problem 1 [4 points]: Distributional Derivatives
Let

$$g(x) := \begin{cases} x & \text{, for } x \geq 0 \\ 0 & \text{, for } x < 0, \end{cases}$$

and T_g the corresponding distribution. Compute all distributional derivatives of T_g (i.e., the derivative to arbitrary order).

Problem 2 [3 points]: Dilations ctd.
We continue Problem 3 from Homework 2. How does one have to define $\tilde{D}_\sigma^p : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ in order to extend D_σ^p?

Problem 3 [3 points]: Fourier Transform
Compute the Fourier transform of the (first) distributional derivative of the delta distribution.

Problem 4 [4 points]: Cauchy Principal Part
The Cauchy principle part integral is defined as

$$\mathscr{P}\left(\frac{1}{x}\right) : f \mapsto \lim_{\varepsilon \downarrow 0} \int_{|x| \geq \varepsilon} \frac{1}{x} f(x) \, \mathrm{d}x$$

for any $f \in \mathcal{S}(\mathbb{R})$. Show that this is indeed a tempered distribution.

Problem 5 [6 points]: Away from the support of $\widehat{\psi}$
Let $\psi_0 \in \mathcal{S}$ and let its Fourier transform have compact support, i.e., $K = \operatorname{supp}(\widehat{\psi})$ is compact. Let U be an open ε-neighborhood of K, i.e., the distance between the complement of U and K is ε, i.e., $\operatorname{dist}(U^c, K) = \varepsilon > 0$. Prove that then for any $m \in \mathbb{N}$ there is a constant $C_{m,\varepsilon}$ such that for any t, x with $x/t \notin U$ and $|t| \geq 1$,

$$\left| \left(e^{-it(-\Delta)/2} \psi_0 \right)(t, x) \right| \leq C_{m,\varepsilon} \left(1 + |t|\right)^{-m}.$$

Hint: One could write the phase factor as $e^{i\alpha S}$ with $S(k) = \frac{kx - k^2 t/2}{1+|t|}$ and some α. Prove that

$$e^{i\alpha S(k)} = \left[\frac{1}{i\alpha} |(\nabla S)(k)|^{-2} (\nabla S)(k) \nabla \right]^m e^{i\alpha S(k)},$$

and then integrate by parts.