Jacobs University
Fall 2021

October 15, 2021

Foundations of Mathematical Physics

Homework 5

Due on October 22, 2021

Problem 1 [4 points]: Uncertainty on $\mathcal{S}$
Prove the Heisenberg uncertainty principle. Let

$$(\delta x_j)^2 := \langle \psi, \big(x_j - \langle \psi, x_j \psi\rangle\big)^2 \psi\rangle, \qquad (\delta p_j)^2 := \langle \psi, \big(p_j - \langle \psi, p_j \psi\rangle\big)^2 \psi\rangle$$

be the variances in the position and the asymptotic momentum distributions, where $p_j = -i\partial_{x_j}$ and $\langle f, g\rangle = \int \overline{f} g$. (Let's just take $\psi \in \mathcal{S}$ here.) Prove that

$$\delta x_j \delta p_j \geq \frac{\|\psi\|^2}{2}.$$

Problem 2 [4 points]: Refined Uncertainty on $\mathcal{S}$
Prove the refined uncertainty principle (Hardy's inequality) on $\mathcal{S}(\mathbb{R}^3)$, i.e., that

$$\langle \psi, (-\Delta)\psi\rangle \geq \frac{1}{4}\langle \psi, |x|^{-2}\psi\rangle$$

for all $\psi \in \mathcal{S}(\mathbb{R}^3)$. *Hint: Look at the quantity $\big[|x|^{-1}p_j|x|^{-1}, x_j\big]$, where $[A, B] := AB - BA$ is the commutator. Note: This could be directly used to give a proof of the stability of hydrogen atoms.*

Problem 3 [12 points]: Cauchy Principal Value continued
Recall from Homework Sheet 4 that the Cauchy principle part

$$\mathscr{P}\left(\frac{1}{x}\right) : \mathcal{S} \to \mathbb{C} : f \mapsto \lim_{\varepsilon \downarrow 0} \int_{|x| \geq \varepsilon} \frac{1}{x} f(x)\, \mathrm{d}x$$

is a tempered distribution.

(a) Prove that

$$\lim_{\varepsilon \downarrow 0} \frac{x - x_0}{(x - x_0)^2 + \varepsilon^2} = \mathscr{P}\left(\frac{1}{x - x_0}\right),$$

in the weak* sense.

(b) Let (φ_n) be a sequence of bounded functions on $\mathbb{R}$ so that $\int_{|x-x_0|\geq\varepsilon} \varphi_n(x)\, \mathrm{d}x \to 0$ as $n \to \infty$ for each $\varepsilon > 0$, $\varphi_n(x) \geq 0$, and $\int \varphi_n(x)\, \mathrm{d}x = c$ independent of n. Prove that $\varphi_n \to c\delta(x - x_0)$ in the weak* sense (i.e., φ_n is here regarded as a distribution).

(c) Prove that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{(x - x_0)^2 + \varepsilon^2} = \pi\delta(x - x_0)$$

in the weak* sense.

(d) Prove the formula

$$\lim_{\varepsilon \downarrow 0} \frac{1}{x - x_0 + i\varepsilon} = \mathscr{P}\left(\frac{1}{x - x_0}\right) - i\pi\delta(x - x_0).$$

(e) Compute the Fourier transform of $\mathscr{P}\left(\frac{1}{x}\right)$. (*Hint: Use part (d).*)