Jacobs University | October 22, 2021
Fall 2021

Foundations of Mathematical Physics

Homework 6

Due on October 29, 2021

Problem 1 [5 points]: Orthonormal basis
Prove that an orthonormal sequence $(\varphi_j)_j$ in a Hilbert space is an orthonormal basis if and only if

$$\langle \varphi_j, \psi \rangle = 0 \quad \text{for all } j \in \mathbb{N} \quad \Rightarrow \quad \psi = 0.$$

Problem 2 [7 points]: Operator norm
Prove the following lemma that was stated in class: Let $\mathcal{L}(X,Y)$ be the set of bounded linear operators from $X \to Y$. Then $\mathcal{L}(X,Y)$ with the norm

$$\|L\|_{\mathcal{L}(X,Y)} := \sup_{\|x\|_X = 1} \|Lx\|_Y$$

is a normed space. Furthermore, if Y is a Banach space, then so is $\mathcal{L}(X,Y)$.

Problem 3 [8 points]: Fourier transform

(a) Let $f \in C_c^\infty(\mathbb{R}^d)$, and let $0 < \alpha < d$. Prove that then

$$c_\alpha \mathcal{F}^{-1}(|k|^{-\alpha} \widehat{f}(k))(x) = c_{d-\alpha} \int |x-y|^{\alpha-d} f(y)\, \mathrm{d}y$$

for some constant c_α, and determine c_α explicitly. *Hint:* $\int_0^\infty e^{-\pi k^2 \lambda} \lambda^{\alpha/2-1}\, \mathrm{d}\lambda$.

Note: In this sense we can give a meaning to the Fourier transform of $|x|^{\alpha-d}$.

(b) The function $g(x) = |x|^{\alpha-d}$ is not in $L^1(\mathbb{R}^d)$. Other than above, how and why exactly can we define the Fourier transform of g?