Jacobs University October 29, 2021
Fall 2021

Foundations of Mathematical Physics

Homework 7

Due on November 5, 2021

Problem 1 [8 points]: Riemann–Lebesgue lemma

(a) Prove that

$$C_\infty(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) : \lim_{R\to\infty} \sup_{|x|>R} |f(x)| = 0\}$$

with the supremum norm is a Banach space. You can use that the space of continuous bounded functions $C_b(\mathbb{R}^d)$ with the supremum norm is a Banach space.

(b) Now prove the Riemann–Lebesgue lemma, i.e., prove that $\mathcal{F}L^1(\mathbb{R}^d) \subset C_\infty(\mathbb{R}^d)$. In order to do so you could first prove continuity of $\mathcal{F} : (\mathcal{S}(\mathbb{R}^d), \|\cdot\|_{L^1(\mathbb{R}^d)}) \to (C_\infty(\mathbb{R}^d), \|\cdot\|_\infty)$. Then you could use that $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)$ and Theorem 3.20 from class in order to continuously extend $\mathcal{F}$ in $L^1(\mathbb{R}^d)$. Why does this extension agree with the usual formulas for $\mathcal{F}$ on all of $L^1(\mathbb{R}^d)$?

Problem 2 [6 points]: Sequences of operators
Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with an orthonormal basis $(\varphi_n)_n$. Give one example (with proof, for (a) and (b) separately), of a sequence $(A_n)_n$ in $\mathcal{L}(\mathcal{H})$ and an A in $\mathcal{L}(\mathcal{H})$, such that

(a) A_n converges weakly to A but not strongly;

(b) A_n converges strongly to A but not in norm.

Problem 3 [6 points]: Sobolev Lemma
For $m \in \mathbb{Z}$, we define the m-th Sobolev space $H^m(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ as the set of $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\widehat{f}$ is measurable and $(1+|k|^2)^{m/2}\widehat{f} \in L^2(\mathbb{R}^d)$. Now let $\ell \in \mathbb{N}_0$ and $f \in H^m(\mathbb{R}^d)$ with $m > \ell + \frac{d}{2}$. Prove that then $f \in C^\ell(\mathbb{R}^d)$ and $\partial^\alpha f \in C_\infty(\mathbb{R}^d)$ for all $|\alpha| \le \ell$. *Hint: Riemann–Lebesgue and Cauchy–Schwarz.*