

# Foundations of Mathematical Physics

## Homework 10

Due on November 26, 2021

### Problem 1 [4 points]: Operator bounds

Let  $A, B$  be densely defined linear operators with  $D(A) \subset D(B)$ . Prove that the following two statements are equivalent:

(i) There are  $a, b \geq 0$  such that

$$\|B\varphi\| \leq a \|A\varphi\| + b \|\varphi\|$$

for all  $\varphi \in D(A)$ .

(ii) There are  $\tilde{a}, \tilde{b} \geq 0$  such that

$$\|B\varphi\|^2 \leq \tilde{a}^2 \|A\varphi\|^2 + \tilde{b} \|\varphi\|^2$$

for all  $\varphi \in D(A)$ .

Prove also that the infimum over all permissible  $a$  in (i) coincides with the infimum over all permissible  $\tilde{a}$  in (ii).

### Problem 2 [8 points]: Self-adjointness

Let  $V : \mathbb{R}^3 \rightarrow \mathbb{R}$  be such that  $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ . Prove that then  $V$  is infinitesimally  $H_0$ -bounded, where  $H_0 = -\frac{1}{2}\Delta$  with domain  $D(H_0) = H^2(\mathbb{R}^3)$ . With Kato-Rellich, this implies that  $H = H_0 + V$  is self-adjoint on  $D(H_0)$ . *Hint: Prove that for all  $a > 0$  there is a  $b > 0$  such that for all  $\varphi \in H^2(\mathbb{R}^3)$  we have  $\|\varphi\|_{L^\infty} \leq a \|\Delta\varphi\|_{L^2} + b \|\varphi\|_{L^2}$ .*

### Problem 3 [8 points]: Projectors

For  $\varphi \in L^2(\mathbb{R}^3)$  with  $\|\varphi\| = 1$ , we define for any  $0 \leq k \leq N$  the bounded operator

$$P_{N,k}^\varphi := \sum_{\mathbf{a} \in \mathcal{A}_k} \prod_{j=1}^N (p_j^\varphi)^{1-a_j} (q_j^\varphi)^{a_j},$$

where

$$\mathcal{A}_k := \left\{ \mathbf{a} \in \{0, 1\}^N : \sum_{j=1}^N a_j = k \right\}.$$

Prove the following properties:

- (a)  $P_{N,k}^\varphi$  is an orthogonal projector for all  $0 \leq k \leq N$ ,
- (b)  $P_{N,k}^\varphi P_{N,j}^\varphi = 0$  for all  $j \neq k$ ,
- (c)  $\sum_{k=0}^N P_{N,k}^\varphi = 1$ ,
- (d)  $\sum_{k=0}^N \frac{k}{N} P_{N,k}^\varphi = \frac{1}{N} \sum_{j=1}^N q_j^\varphi$ , with  $q_j^\varphi$  as defined in class.