We continue our remarks about $\Psi(x_1,\dots,x_N)$:

- fact: for $d=3$, wave function is either
  - $\boxed{\text{bosonic}}$, meaning symmetric under exchange of variables, i.e.,

    $$\Psi(\dots,x_j,\dots,x_k,\dots) = \Psi(\dots,x_k,\dots,x_j,\dots)$$

    symmetric group (i.e., permutations of $\{1,\dots,N\}$)

    $$\left(\text{i.e., } \Psi(x_1,\dots,x_N) = \Psi(x_{\sigma(1)},\dots,x_{\sigma(N)})\ \forall \sigma\in S_N\right)$$

  - or $\boxed{\text{fermionic}}$, meaning antisymmetric under exchange of variables, i.e.,

    $$\Psi(\dots,x_j,\dots,x_k,\dots) = -\Psi(\dots,x_k,\dots,x_j,\dots)$$

    $$\left(\text{i.e., } \Psi(x_1,\dots,x_N) = \underbrace{(-1)^\sigma}_{=\,\mathrm{sgn}(\sigma)\,=\,\text{sign of permutation }\sigma\,=\,\begin{cases}+1 & \text{for } \sigma \text{ even}\\ -1 & \text{for } \sigma\text{ odd}\end{cases}} \Psi(x_{\sigma(1)},\dots,x_{\sigma(N)})\ \forall\sigma\in S_N\right)$$

Note:
- only particles of the same kind have these symmetries
  (e.g., x_1,x_2 bosons with mass m, y_1,y_2 bosons with mass $\tilde{m}\neq m$, z_1,z_2 fermions, then
  $\Psi(x_1,x_2,y_1,y_2,z_1,z_2)$ symm. in x_1,x_2, symm. in y_1,y_2, antisymm. in z_1,z_2)
- bosons: "tend to be in the same state" (see HW 1), which, e.g., leads to Bose-Einstein condensation
- fermions: "tend to repel each other" (HW 1), which, e.g., leads to the Fermi pressure (neutron stars, ...), superconductivity etc.

- reason for symm./antisymm. is that really $\,^N\mathbb{R}^d := \overbrace{\{q \subset \mathbb{R}^d : |q| = N\}}^{N \text{ element subsets of } \mathbb{R}^d}$ is the right configuration space, and not $\mathbb{R}^{dN}$ ("all particles are indistinguishable"), for the same particle species; also, we need

  $|\psi(x_1, \dots, x_N)|^2 = |\psi(x_{\sigma(1)}, \dots, x_{\sigma(N)})|^2$

- $\,^N\mathbb{R}^d$ has interesting topology: its connectedness properties lead to the different symmetries
  - ↳ $\,^N\mathbb{R}^3$ is multiply connected, which leads to the boson-fermion alternative
  - ↳ $\,^N\mathbb{R}^2$ is multiply connected "in a worse way", which leads to many more possibilities: anyons, with $\psi(\dots, x_j, \dots, x_k, \dots) = e^{i\pi\alpha}\psi(\dots, x_k, \dots, x_j, \dots)$ (relevant for quasi 2 dim. materials)

- also interesting/relevant is the time-independent Schrödinger equation: $\boxed{H\phi_E = E\phi_E}$ (an eigenvalue problem).

  $E \in \mathbb{R}$

  Hamiltonian   wave function

Here, E = eigenvalues/energies ; ϕ_E = eigenfunctions/eigenstates

↳ give solution $\psi(t) = e^{-iEt}\phi_E$ to time-dependent Schrödinger equation

(check: $i\partial_t \psi(t) = i\partial_t e^{-iEt}\phi_E = e^{-iEt}E\phi_E = e^{-iEt}H\phi_E = H\psi(t)$ ✓)

↳ $E_0 = \inf\{E\} = \inf\limits_{\phi, \|\phi\|=1} \langle\phi, H\phi\rangle$, if it exists, is called ground state energy;

if a minimizer ϕ_{E_0} exists, it is called ground state.

Ground states are very relevant, since matter tends to radiate until it reaches lowest energy state

↳ ϕ_E with $E > E_0$ are called excited states

↳ active research topics: find/approximate: • E_0, Φ_{E_0}

• low lying E, Φ_E

(• all E, Φ_E rarely possible)

increasing level of difficulty

• Finally, let us write down the Hamiltonian of non-relativistic matter for N electrons. We treat the nuclei in Born-Oppenheimer approximation (i.e., "classically"), i.e., they are at positions $Y_1(t), \dots, Y_M(t)$ ($Y_j(t) \in \mathbb{R}^3$); they have charges $Z_1, \dots, Z_M$ ($Z_j \in \mathbb{Z}$, i.e., multiples of the elementary charge e). Denoting the electron variables as $x_1, \dots, x_N$ (i.e., the wave function is $\Psi_t(x_1, \dots, x_N)$), the Hamiltonian is:

$$\boxed{H = \sum_{j=1}^{N} \frac{\hbar^2}{2m_e}(-\Delta_{x_j}) + \hbar c \alpha \left(\underbrace{\sum_{1 \le j < k \le M} \frac{Z_j Z_k}{|Y_j(t) - Y_k(t)|}}_{= \text{const}(t) = \text{energy of nuclei}} - \underbrace{\sum_{j=1}^{N} \sum_{k=1}^{M} \frac{Z_k}{|x_j - Y_k(t)|}}_{= \text{attractive external field of nuclei}} + \underbrace{\sum_{1 \le j < k \le N} \frac{1}{|x_j - x_k|}}_{= \text{repulsive Coulomb interaction of electrons}} \right)}$$

↳ m_e = electron mass

↳ note: for a neutral molecule: $\sum_{j=1}^{M} Z_j = N$

This Hamiltonian describes – sometimes with small modifications – non relativistic matter (e.g., chemistry, conductivity, ...)

Central topic of this class:

For which $\psi(t=0)$ and V does Schrödinger equation have global solutions, and in which sense?

general idea: regard Schrödinger equation as an ODE $i\frac{d}{dt}\psi(t) = H\psi(t)$ for

$$\psi : \mathbb{R} \to \mathcal{H} = \text{some Hilbert space, usually } L^2(\mathbb{R}^{dN})$$

difficulty:
- $\mathcal{H}$ infinite dimensional
- H unbounded