Recall: we defined $L^p(\mathbb{R}^d) := \left\{ f: \mathbb{R}^d \to \mathbb{C} : \|f\|_p := \left(\int |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\}$

Remarks:

- $L^\infty(\mathbb{R}^d) := \left\{ f: \mathbb{R}^d \to \mathbb{C} : \|f\|_\infty := \overbrace{\inf\{C \geq 0 : |f(x)| \leq C \text{ for almost all } x\}}^{=: \operatorname{ess\,sup} f \text{ (essential supremum)}} < \infty \right\}$

  Note: one can show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$ $\left(\forall f \in L^\infty \cap L^q \text{ for some } q\right)$

- For all $1 \leq p \leq \infty$, $L^p(\mathbb{R}^d)$ are Banach spaces (i.e., complete normed vector spaces) if one identifies functions that agree almost everywhere (always assumed)
  $\Rightarrow$ really, $L^p(\mathbb{R}^d)$ are vector spaces of equivalence classes of functions

- Only $L^2(\mathbb{R}^d)$ is a $\underbrace{\text{Hilbert space}}_{\text{= Banach space with norm given by scalar product, i.e., } \|f\|^2 = \langle f,f\rangle}$ with scalar product $\langle f,g\rangle = \int \bar{f} g$

Now we can define the Fourier transform on L^1:

Def. 2.1: Let $f, g \in L^1(\mathbb{R}^d)$, then we define the

- Fourier transform of f as $\boxed{\hat{f}(k) = (\mathcal{F}f)(k) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x) e^{-ikx} dx}$,
- inverse Fourier transform of g as $\boxed{\check{g}(x) = (\mathcal{F}^{-1}g)(x) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} g(k) e^{ikx} dk}$.

(Note: We do not know yet in what sense $\mathcal{F}^{-1}$ is the inverse of $\mathcal{F}$.)

Next: we want to know about regularity of $\hat{f}$ (i.e., continuity, differentiability)

$\rightarrow$ need to take derivative of integral with parameter

Lemma 2.2: Integrals with Parameter

Let $I(y) = \int_{\mathbb{R}^d} f(x,y)\,dx$, with $f: \mathbb{R}^d \times \Gamma \to \mathbb{C}$, where $\Gamma \subset \mathbb{R}$ an open interval, and let $f(x,y) \in L^1(\mathbb{R}^d)$ for all fixed $y \in \Gamma$.

a) If $y \mapsto f(x,y)$ is continuous for $\overbrace{\text{almost all}}^{:=\text{a.a.}}$ $x \in \mathbb{R}^d$

and if $\exists\, g \in L^1(\mathbb{R}^d)$ with $\sup_{y\in\Gamma} |f(x,y)| \leq g(x)$ for a.a. $x \in \mathbb{R}^d$,

then $I(y)$ is continuous.

b) If $y \mapsto f(x,y)$ is continuously differentiable $\forall x \in \mathbb{R}^d$

and if $\exists\, g \in L^1(\mathbb{R}^d)$ with $\sup_{y\in\Gamma} |\partial_y f(x,y)| \leq g(x)$ $\forall x \in \mathbb{R}^d$,

then $I(y)$ is continuously differentiable and

$$\frac{dI(y)}{dy} = \frac{d}{dy}\int_{\mathbb{R}^d} f(x,y)\,dx = \int_{\mathbb{R}^d} \partial_y f(x,y)\,dx.$$

Proof: HW. Use dominated convergence.

(Note: Lemmas like this one are one of the main advantages of Lebesgue over Riemann integral.)

We need to introduce more notation:

- a $\boxed{\text{multi-index}}$ $\alpha \in \mathbb{N}_0^d$ is a typel $(\alpha_1, \dots, \alpha_d)$, $\alpha_j \in \mathbb{N}_0$.
  We denote $|\alpha| := \sum_{j=1}^{d} \alpha_j$, and for $x \in \mathbb{R}^d$, $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $\partial_x^\alpha := \dfrac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}}$.

- $\boxed{C^p(\mathbb{R}^d)} := \{ f : \underbrace{\partial_x^\alpha f \text{ continuous } \forall \text{ multi-indices } \alpha \text{ with } |\alpha| \leq p}_{f \text{ p times continuously differentiable}} \}$

- $\boxed{C^\infty(\mathbb{R}^d)} = \bigcap_{p \in \mathbb{N}} C^p(\mathbb{R}^d)$ = smooth functions (∞ often continuously differentiable)
- $C^0(\mathbb{R}^d) = \boxed{C(\mathbb{R}^d)}$ = continuous functions
- $\underbrace{\boxed{C_\infty(\mathbb{R}^d)}}_{\text{Sometimes called } C_0(\mathbb{R}^d)} := \{ f \in C(\mathbb{R}^d) : \underbrace{\lim_{|x| \to \infty} f(x) = 0}_{\text{more exact: } \lim_{R \to \infty} \sup_{|x| > R} |f(x)| = 0} \}$

- $\boxed{C_c^p(\mathbb{R}^d)} := C^p(\mathbb{R}^d) \cap \{ f : \underbrace{\operatorname{supp} f}_{\text{support of } f} \ \underbrace{\text{compact}}_{\text{in } \mathbb{R}^d, \text{ compact} \Leftrightarrow \text{closed and bounded}} \}$ = functions with compact support

Where does the Fourier transform on L^1 map to? We know the following:

Lemma 2.3: Riemann-Lebesgue

$$f \in L^1(\mathbb{R}^d) \Rightarrow \hat{f} \in C_\infty(\mathbb{R}^d)$$

Proof: $\cdot$ $f \in L^1(\mathbb{R}^d)$, recall $\hat{f}(k) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \underbrace{f(x) e^{-ikx}}\, dx$

↳ cont. in k for a.a. x

↳ $\sup_k |f(x) e^{-ikx}| = |f(x)| \in L^1(\mathbb{R}^d)$

$\Rightarrow \hat{f}$ continuous with Lemma 2.2.

$\cdot$ decay at ∞ relatively easy to show, follows later from a more general result. $\square$

Next: a class of very nice functions where we can define the Fourier transform as a bijection.

Definition 2.5: Schwartz space

We call the $\mathbb{C}$-vector space

$$S(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) : \|f\|_{\alpha,\beta} < \infty \ \forall \text{ multi-indices } \alpha,\beta \in \mathbb{N}_0^d \right\}$$ Schwartz space

(space of smooth rapidly decaying functions). Here,

$$\|f\|_{\alpha,\beta} := \|x^\alpha \partial_x^\beta f(x)\|_\infty = \sup_{x\in\mathbb{R}^d} |x^\alpha \partial_x^\beta f(x)| .$$

Next time: construct metric on S, and properties of $\mathcal{F}$ on S.