Last time we introduced the Schwartz space

$$S(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) : \|f\|_{\alpha,\beta} < \infty \ \forall \text{ multi-indices } \alpha,\beta \in \mathbb{N}_0^d \right\}, \text{ with}$$

$$\|f\|_{\alpha,\beta} := \|x^\alpha \partial_x^\beta f(x)\|_\infty = \sup_{x\in\mathbb{R}^d} |x^\alpha \partial_x^\beta f(x)|.$$

Note: • for $f \in S(\mathbb{R}^d)$, f and all partial derivatives decay faster than any polynomial

• e.g., $e^{-x^2} \in S(\mathbb{R}^d)$, $C_c^\infty(\mathbb{R}^d) \subset S(\mathbb{R}^d)$

Definition: On a vector space V, a map $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ is called semi-norm if

- $\|\lambda f\| = |\lambda| \cdot \|f\|$ (absolute homogeneity)
- $\|f+g\| \leq \|f\| + \|g\|$ (triangle inequality)

Note: • for a norm, we require additionally that $\|f\| = 0 \Rightarrow f = 0$

• $\|f\|_{\alpha,\beta}$ are semi-norms (for $\beta = 0$, $\|f\|_{\alpha,0}$ is also a norm)

↳ e.g., $d=1$, $\|x\|_{0,2} = \|\partial_x^2 x\|_\infty = 0$ (but $f(x) = x \not\equiv 0$)

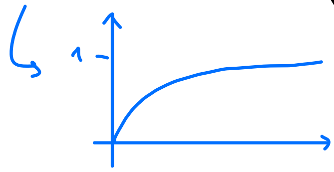
Next: since we have only a family of semi-norms on S, it is not a Banach space; but we can construct a complete metric space (in this context called a Fréchet space) in the following way.

Lemma 2.8:

$$d_S(f,g) := \sum_{n=0}^\infty 2^{-n} \sup_{|\alpha|+|\beta|=n} \left(\frac{\|f-g\|_{\alpha,\beta}}{1+\|f-g\|_{\alpha,\beta}} \right) \text{ is a metric on } S.$$

Note: the choice of $\frac{\|\cdots\|_{\alpha,\beta}}{1+\|\cdots\|_{\alpha,\beta}}$ is a convention; we could choose other functions that lead to the triangle inequality and go to zero for $\|\cdots\|_{\alpha,\beta}$ going to zero.

<u>Proof:</u> First, note that $\frac{x}{1+x}$ maps $\mathbb{R}_{\geq 0}$ to $[0,1]$ and is monotonically increasing.



$$\Rightarrow d_S(f,g) \leq \sum_{n=0}^{\infty} 2^{-n} = \frac{1}{1-\frac{1}{2}} = 2$$

We now check the properties of a metric:

- $d_S(f,g) \geq 0$ clear

- $d_S(f,g) = d_S(g,f)$ clear

- $d_S(f,g) = 0 \Leftrightarrow f = g$?

  ↳ "$\Leftarrow$" clear

  ↳ "$\Rightarrow$" let $d_S(f,g) = 0$;

  then in particular $\|f-g\|_{0,0} = \|f-g\|_\infty := \sup_{x\in\mathbb{R}^d} |f(x)-g(x)| = 0 \Rightarrow f = g$

- $d_S(f,g) \leq d_S(f,h) + d_S(h,g)$?

  ↳ we have $\|f-g\|_{\alpha,\beta} = \|f-h+h-g\|_{\alpha,\beta} \leq \underbrace{\|f-h\|_{\alpha,\beta}}_{:=x} + \underbrace{\|h-g\|_{\alpha,\beta}}_{=:y}$

  ↳ then $\frac{\|f-g\|_{\alpha,\beta}}{1+\|f-g\|_{\alpha,\beta}} \overset{\text{monotone increasing}}{\leq} \frac{x+y}{1+x+y} = \frac{x}{1+x+y} + \frac{y}{1+x+y} \leq \frac{x}{1+x} + \frac{y}{1+y}$ ✓ $\square$

$$\boxed{\begin{array}{l}\underline{\text{Corollary:}}\ \text{Convergence in } S \\[1em] f_n \xrightarrow{n\to\infty} f \text{ in } S :\Leftrightarrow d_S(f,f_n) \xrightarrow{n\to\infty} 0 \\[1em] \qquad\qquad \Leftrightarrow \|f-f_n\|_{\alpha,\beta} \xrightarrow{n\to\infty} 0 \quad \forall \alpha,\beta\in\mathbb{N}_0^d.\end{array}}$$

An important property is:

$\boxed{\underline{\text{Lemma 2.9}}\text{: The metric space } (S, d_S) \text{ is complete.}}$

Recall: • $(f_m)_m$ is a Cauchy sequence means: $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $d(f_m, f_n) < \varepsilon\ \forall m, n > N$

• Clearly every convergent sequence is also a Cauchy sequence since
$d(f_m, f_n) \leq d(f_m, f) + d(f_n, f)$ (if RHS $\to 0$ then also LHS $\to 0$)

• In the definition of a Cauchy sequence we only use the f_m (not a possible limit f); this is technically nice and often easier to work with. If completeness holds (i.e., $(f_m)_m$ Cauchy $\Leftrightarrow$ $(f_m)_m$ converges), we just have to check the Cauchy property and then know that a limit always exists.

<u>Proof:</u> Let $(f_m)_m$ be a Cauchy sequence in S.

Idea: We first construct a candidate f for the limit, and then show that it is indeed the limit in S.

Note: $(f_m)_m$ Cauchy in S $\Rightarrow$ (f_m) is also Cauchy w.r.t. all $\|\cdot\|_{\alpha,\beta}$;

put differently: $f_m^{(\alpha,\beta)}(x) := x^\alpha \partial_x^\beta f_m(x)$ is Cauchy w.r.t. $\|\cdot\|_\infty$

From Analysis II we (should) know that $C_b(\mathbb{R}^d) := \{ f \in C(\mathbb{R}^d) : f \text{ bounded} \}$ is complete w.r.t. $\|\cdot\|_\infty$. Thus $f_m^{(\alpha,\beta)} \xrightarrow{m\to\infty} f^{(\alpha,\beta)}$ uniformly.

(See, e.g., Rudin: Principles of Mathematical Analysis (3rd edition) Theorem 7.15)

Therefore, $f := f^{(0,0)}$ is the candidate for the limit of $(f_m)_m$. But so far we only know $f^{(0,0)} \in C_b$. We need to show: $f \in C^\infty(\mathbb{R}^d)$ and $x^\alpha \partial_x^\beta f(x) = f^{\alpha,\beta}(x)$.
This would imply $f \in S(\mathbb{R}^d)$ and $d_S(f_m, f) \xrightarrow{m\to\infty} 0$, i.e., $f_m \xrightarrow{m\to\infty} f$ in S, and thus the completeness of (S, d_S).

Checking this in detail is a bit lengthy; let us here just show for $d=1$ that $f \in C^1(\mathbb{R}^d)$ and $\partial_x f = f^{(0,1)}$, the rest goes analogously.

Since $f_m \in \mathcal{S}(\mathbb{R})$ $\forall m$, we have $f_m(x) = f_m(0) + \int_0^x f_m'(y)\,dy$.

Since $f_m \to f$ and $f_m' \to f^{(0,1)}$ <u>uniformly</u>, we can take the limit:

$$\lim_{m\to\infty} f_m(x) = f(x) = f(0) + \underbrace{\lim_{m\to\infty} \int_0^x f_m'(y)\,dy}$$

$$= \int_0^x f^{(0,1)}(y)\,dy \quad \text{due to uniform convergence}$$

Thus, $f \in C^1(\mathbb{R})$ and $f' = f^{(0,1)}$ $\square$