Today we establish some standard properties of the Fourier transform on $\mathcal{S}$.

<u>Lemma 2.11</u>: Properties of the Fourier transform

(1) $\forall \alpha, \beta \in \mathbb{N}_0^d$, $f \in \mathcal{S}$: $\left((ik)^\alpha \partial_k^\beta \mathcal{F} f \right)(k) = \left(\mathcal{F} \partial_x^\alpha (-ix)^\beta f \right)(k)$,

in particular: $\widehat{(xf)}(k) = i (\nabla_k \hat{f})(k)$ and $\widehat{(\nabla_x f)}(k) = ik \hat{f}(k)$.

(2) $\mathcal{F}$ and $\mathcal{F}^{-1}$ are continuous linear maps $\mathcal{S} \to \mathcal{S}$.

<u>Proof</u>:

(1) recall $(\mathcal{F} f)(k) = \hat{f}(k) = (2\pi)^{-\frac{d}{2}} \int f(x) e^{-ikx} dx$

Then with Lemma 2.2: $\left((ik)^\alpha \partial_k^\beta \mathcal{F} f \right)(k) = (2\pi)^{-\frac{d}{2}} \int (ik)^\alpha \partial_k^\beta e^{-ikx} f(x) dx$

note: all integrals exist since $f \in \mathcal{S}$

$= (2\pi)^{-\frac{d}{2}} \int (ik)^\alpha (-ix)^\beta e^{-ikx} f(x) dx$

$= (2\pi)^{-\frac{d}{2}} (-1)^{|\alpha|} \int (\partial_x^\alpha e^{-ikx}) (-ix)^\beta f(x) dx$

$|\alpha|$-times integration by parts (boundary terms vanish, since $f \in \mathcal{S}$)

$= (2\pi)^{-\frac{d}{2}} \int e^{-ikx} \left(\partial_x^\alpha (-ix)^\beta f(x) \right) dx$

$= \mathcal{F}\left(\partial_x^\alpha (-ix)^\beta f \right)(k)$

(2) On metric spaces continuity (preimages of open sets are open) is equivalent to sequential continuity ($d(f_n,f)\to 0$ implies $d(\mathcal{F}f_n,\mathcal{F}f)\to 0$).

Thus let us choose $f_n \to f$ in $\mathcal{S}$, meaning $d_{\mathcal{S}}(f_n,f)\to 0$, meaning $\|f_n - f\|_{\alpha,\beta}\to 0$ for all $\alpha,\beta\in\mathbb{N}_0^d$. We now show that $\|\mathcal{F}g\|_{\alpha,\beta}\le C\sum_{j=0}^{m}\sup_{|\tilde\alpha|+|\tilde\beta|=j}\|g\|_{\tilde\alpha,\tilde\beta}$ for some $C>0$ and $m\in\mathbb{N}$, which implies $\|\mathcal{F}f-\mathcal{F}f_n\|_{\alpha,\beta}\to 0$ $\forall\alpha,\beta\in\mathbb{N}_0^d$ if $\|f_n-f\|\to 0$ $\forall \alpha,\beta\in\mathbb{N}_0^d$.

We compute:

$$\|\mathcal{F}g\|_{\alpha,\beta} = \|k^\alpha\partial_k^\beta \mathcal{F}g\|_\infty$$

(1) and $|\int\ldots|\le\int|\ldots|$

$$\le (2\pi)^{-\frac{d}{2}}\int |\partial_x^\alpha x^\beta g(x)|\,dx$$

$$= (2\pi)^{-\frac{d}{2}}\int (1+|x|^2)^d\,|\partial_x^\alpha x^\beta g(x)|\,(1+|x|^2)^{-d}\,dx$$

$$\le (2\pi)^{-\frac{d}{2}}\left(\sup_{x\in\mathbb{R}^d}(1+|x|^2)^d|\partial_x^\alpha x^\beta g(x)|\right)\underbrace{\int(1+|x|^2)^{-d}\,dx}_{=\text{const}\cdot\int_0^\infty(1+r^2)^{-d}r^{d-1}\,dr\,\le\,\tilde C}$$

(since integrand $\sim r^{-2d+d-1}$ for large r)

$$\le C\sum_{j=0}^{m}\sup_{|\tilde\alpha|+|\tilde\beta|=j}\|g\|_{\tilde\alpha,\tilde\beta}\quad\text{for some } m\in\mathbb{N},\ C>0.$$

□

Theorem 2.12:

$\mathcal{F}:\mathcal{S}\to\mathcal{S}$ is a continuous bijection with continuous inverse $\mathcal{F}^{-1}$.

Proof: HW: (1) Show $\mathcal{F}^{-1}\mathcal{F}=\mathrm{id}$ only on C_c^∞ = smooth fct.s with compact support

↳ consider $\operatorname{supp} f\subset[-m,m]^d$

$\Rightarrow$ Fourier series, write f as Riemann sum

(2) Show that C_c^∞ is dense in $\mathcal{S}$, then thm. follows from continuity.

use some smooth cutoff function, e.g., $G(x) = \begin{cases} e^{-\frac{1}{1-x^2}+1} & \text{for } x<1 \\ 0 & \text{else.} \end{cases}$

Lemma 2.14: Plancherel on $\mathcal{S}$

For $f, g \in \mathcal{S}$, we have $\int \hat{f}(x) g(x)\,dx = \int f(x)\,\hat{g}(x)\,dx$, and, in particular,

$$\int |\hat{f}(k)|^2\,dk = \int |f(x)|^2\,dx.$$

Proof: simple computation, HW.

Now back to the free SE $\;i\partial_t \psi(t,x) = -\frac{1}{2}\Delta_x \psi(t,x)$

Formally we solve this by applying $\mathcal{F}$: $\;i\partial_t \hat{\psi}(t,k) = -\frac{1}{2}(\mathcal{F}\Delta_x\psi)(t,k) \overset{\text{Lemma 2.11}}{=} \frac{1}{2}k^2\hat{\psi}(t,k)$

$\Rightarrow \hat{\psi}(t,k) = e^{-i\frac{k^2}{2}t}\,\hat{\psi}(0,k)$ unique global solution

$\Rightarrow \psi(t,x) = \left(\mathcal{F}^{-1} e^{-i\frac{k^2}{2}t}\,\mathcal{F}\psi_0\right)(x)$, with $\psi_0(x) = \psi(0,x)$ the initial condition

As a map $\psi: \mathbb{R} \to \mathcal{S}$, this is indeed a differentiable solution to the free SE (even ∞-often differentiable). (Theorem next time.)