Last time: $\psi(t,x) = \left(\mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F}\psi_0\right)(x)$ formally solves the free SE.

More exactly:

Theorem 2.16: Solution to free SE in S

Let $\psi_0 \in S(\mathbb{R}^d)$. Then the unique $\overbrace{\text{global}}^{\text{for all } t \text{ (as opposed to "local" = for some finite time interval)}}$ solution $\psi \in C^\infty(\mathbb{R}_t, S(\mathbb{R}^d))$ to the free SE with $\psi(0,x) = \psi_0(x)$ is, for $t \neq 0$,

$$\psi(t,x) = \left(\mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F}\psi_0\right)(x) = (2\pi i t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{2t}} \psi_0(y)\,dy.$$

Furthermore, $\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^d)} = \|\psi_0\|_{L^2(\mathbb{R}^d)}$ $\forall t \in \mathbb{R}$.

Important note: What does $\boxed{\psi \in C^\infty(\mathbb{R}_t, S(\mathbb{R}^d))}$ mean?

First, ψ is a map from $\mathbb{R}$ to $S(\mathbb{R}^d)$, i.e., for fixed t, $\psi(t,x)$ as a function of x lies in S.

Second, the map $\psi: \mathbb{R}_t \to S(\mathbb{R}^d)$ is ∞-often differentiable, i.e.,

$$\frac{\psi(t+h,\cdot) - \psi(t,\cdot)}{h} \xrightarrow[\text{in } S]{h \to 0} \dot\psi(t) \text{ for some } \dot\psi(t) \in S.$$

Proof: The formula $\psi(t,x) = \left(\mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F}\psi_0\right)(x) = (2\pi i t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{2t}} \psi_0(y)\,dy$

can be checked by direct computation (use Fourier transform of Gaussian).

Next: Let us show that $t \mapsto \psi(t,\cdot)$ is once differentiable, then $\psi \in C^\infty(\mathbb{R}, S)$ follows by repeating the argument.

Guess: derivative is $\dot{\psi}(t,x) = -i\left(\mathcal{F}^{-1} \frac{k^2}{2} e^{-i\frac{k^2}{2}t} \mathcal{F}\psi_0\right)(x)$, which we know is in $S(\mathbb{R}^d)$.

To show: $\lim_{h\to 0} \left\| \frac{\psi(t+h)-\psi(t)}{h} - \dot{\psi}(t) \right\|_{\alpha,\beta} = 0 \quad \forall \alpha, \beta \in \mathbb{N}_0^d$

By continuity of $\mathcal{F}$ (Lemma 2.11), this is equivalent to

$$\lim_{h\to 0} \left\| \mathcal{F}\left(\frac{\psi(t+h)-\psi(t)}{h} - \dot{\psi}(t) \right) \right\|_{\alpha,\beta} = 0 \quad \forall \alpha,\beta \in \mathbb{N}_0^d$$

We compute: $\underbrace{\text{LHS}}$ $= \lim_{h\to 0} \left\| \frac{\hat{\psi}(t+h)-\hat{\psi}(t)}{h} - \hat{\dot{\psi}}(t) \right\|_{\alpha,\beta}$

left-hand side of eq. above

$$= \lim_{h\to 0} \sup_{k\in\mathbb{R}^d} \left| k^\alpha \partial_k^\beta \left(\frac{e^{-i\frac{k^2}{2}(t+h)} - e^{-i\frac{k^2}{2}t}}{h} + i\frac{k^2}{2} e^{-i\frac{k^2}{2}t} \right)(\mathcal{F}\psi_0)(k) \right| = 0,$$

since $\mathcal{F}\psi_0 \in S$ and $e^{-i\frac{k^2}{2}t}$ smooth as a function of k and t.

We compute furthermore:

$$\|\psi(t,\cdot)\|_{L^2}^2 = \int |\psi(t,x)|^2 dx = \int \left| \left(\mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F}\psi_0\right)(x) \right|^2 dx$$

Plancherel (2.14)

$$= \int \left| e^{-i\frac{k^2}{2}t} \mathcal{F}\psi_0(x) \right|^2 dx$$

$$= \int |\mathcal{F}\psi_0(x)|^2 dx$$

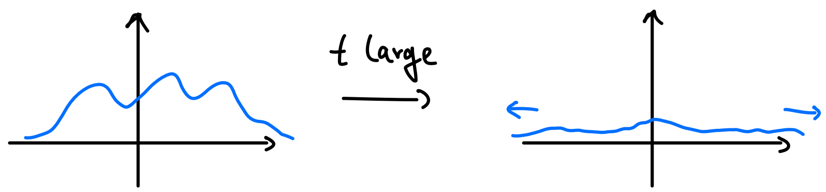
Plancherel (2.14)

$$= \int |\psi_0(x)|^2 dx = \|\psi(0,\cdot)\|_{L^2}^2 . \qquad \square$$

Note: $\|\psi(t,\cdot)\|_\infty = \sup_{x\in\mathbb{R}^d} |\psi(t,x)| = \sup_{x\in\mathbb{R}^d} \left| (2\pi i t)^{-\frac{d}{2}} \int e^{i\frac{|x-y|^2}{2t}} \psi_0(y)\,dy \right|$

$$\leq (2\pi t)^{-\frac{d}{2}} \|\psi_0\|_{L^1} \xrightarrow{t\to\infty} 0$$

$\Rightarrow$ wave functions spread:



Next: A few more properties/applications of the Fourier transform.

We want to define multiplication operators $\psi(x) \mapsto f(x)\psi(x)$ as continuous maps on $\mathcal{S}$, as we did in $e^{-i\frac{k^2}{2}t}\hat{\psi}_0$. For that, f cannot be too wild; an appropriate space is:

Definition 2.18: The space of smooth polynomially bounded functions is

$$C^\infty_{pol}(\mathbb{R}^d) := \left\{ f\in C^\infty(\mathbb{R}^d) : \forall \alpha\in\mathbb{N}_0^d\ \exists n_\alpha\in\mathbb{N} \text{ and } C_\alpha<\infty \text{ s.t. } |\partial_x^\alpha f(x)| \leq C_\alpha (1+|x|^2)^{\frac{n_\alpha}{2}} \right\}$$

Note:
- a common notation is: $(1+|x|^2)^{\frac{1}{2}} =: \langle x\rangle$
- e.g., all polynomials $\in C^\infty_{pol}$, $e^{ikx}\in C^\infty_{pol}$, $e^x \notin C^\infty_{pol}$

Then indeed:

Lemma: For $f\in C^\infty_{pol}(\mathbb{R}^d)$, the multiplication operator $M_f : \mathcal{S}\to\mathcal{S},\ \psi(x)\mapsto f(x)\psi(x)$ is continuous.

Proof: clear: if $\|\psi_n - \psi\|_{\alpha,\beta} \xrightarrow{n\to\infty} 0\ \forall \alpha,\beta$, then also

$$\|M_f(\psi_n-\psi)\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^d} |x^\alpha \partial_x^\beta (f(x)(\psi_n(x)-\psi(x)))| \xrightarrow{n\to\infty} 0 \qquad \square$$

The solution to the free SE can be written as $\mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F} \psi_0 = \mathcal{F}^{-1} M_f \mathcal{F} \psi_0$ for $f(k) = e^{-i\frac{k^2}{2}t}$. Since multiplication in Fourier space = derivatives in x-space, we introduce the following notation for $\mathcal{F}^{-1} M_f \mathcal{F}$:

<u>Definition 2.19</u>:

For $f \in C^\infty_{pol}(\mathbb{R}^d)$ we define the pseudo-differential operator

$$f(-i\nabla): \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d),\ \psi(x) \longmapsto \big(f(-i\nabla_x)\psi\big)(x) = \left(\mathcal{F}^{-1} M_f \mathcal{F} \psi\right)(x) = \left(\mathcal{F}^{-1} f(k) \mathcal{F} \psi\right)(x)$$

Note:
- $f(-i\nabla)$ continuous, since $M_f, \mathcal{F}, \mathcal{F}^{-1}$ continuous
- $f(k) = k^\alpha \Rightarrow f(-i\nabla) = (-i)^{|\alpha|} \partial_x^\alpha$ is the usual differential operator
- Example: semi-relativistic or pseudo-relativistic Schrödinger equation:

$$i\partial_t \psi(t,x) = \underbrace{\sqrt{1-\Delta}}_{\text{pseudo-differential operator}} \psi(t,x)$$