Last time we introduced pseudo-differential operators $f(-i\nabla) = \mathcal{F}^{-1} M_f \mathcal{F}$ as continuous maps from $\mathcal{S}$ to $\mathcal{S}$, for $f \in C^\infty_{pol}$.

Examples:

- translation operator: for $a \in \mathbb{R}^d$, let $T_a(k) = e^{-iak}$ $\Rightarrow T_a \in C^\infty_{pol}$

$$\Rightarrow \text{for } \psi \in \mathcal{S}, \text{ we find } \big(T_a(-i\nabla)\psi\big)(x) = (2\pi)^{-\frac{d}{2}} \int e^{ikx} e^{-iak} \hat{\psi}(k)\,dk$$
$$= (2\pi)^{-\frac{d}{2}} \int e^{ik(x-a)} \hat{\psi}(k)\,dk$$
$$= \psi(x-a)$$

- free propagator: $P_f(k) = e^{-i\frac{k^2}{2}t}$ $\Rightarrow P_f \in C^\infty_{pol}$

$$\Rightarrow \text{solution to free Schrödinger equation is } \psi(t,x) = \left(\mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F}\psi_0\right)(x)$$
$$= \big(P_f(t,-i\nabla)\psi_0\big)(x)$$

Thus, we found: $\psi(t) = e^{-i\frac{(-\Delta)}{2}t}\psi(0)$

- heat equation: $\partial_t f(t,x) = \frac{1}{2}\Delta_x f(t,x)$

$$\Rightarrow W(t,k) = e^{-\frac{k^2}{2}t} \in C^\infty_{pol} \text{ for } t \geq 0$$

$$\Rightarrow \text{for } f(0,\cdot) = f_0 \in \mathcal{S},\ t>0, \text{ we have } f(t) = e^{\frac{1}{2}\Delta t} f_0 = W(t,-i\nabla) f_0$$

Lastly, $\mathcal{F}^{-1} M_f \mathcal{F} \psi_0 = \mathcal{F}^{-1}(f(k)\hat{\psi}_0(k))$, so we want to know about the (inverse) Fourier transform of a product.

Definition 2.22:

The convolution of $f \in S$ and $g \in S$ is $(f * g)(x) := \int_{\mathbb{R}^d} f(x-y)g(y)\,dy$.

Lemma 2.23: For $f, g, h \in S$ we have

a) $(f*g)*h = f*(g*h)$ and $f*g = g*f$;

b) the map $S \to S$, $g \mapsto f*g$ is continuous;

c) $\widehat{f*g} = (2\pi)^{\frac{d}{2}} \hat{f}\cdot\hat{g}$ and $\widehat{f g} = (2\pi)^{-\frac{d}{2}} \hat{f}*\hat{g}$,

in particular $g(-i\nabla)f = \mathcal{F}^{-1} M_g \mathcal{F} f = \mathcal{F}^{-1} g\hat{f} = (2\pi)^{-\frac{d}{2}} \check{g}*f$.

Proof: • a) and c) are direct calculations

• then b) follows since $f*g = (2\pi)^{\frac{d}{2}} \mathcal{F}^{-1} \hat{f}\hat{g}$, i.e., composition of continuous maps □

Example: heat equation: $f(t,x) = W(t,-i\nabla) f(0,x)$, $W(t,k) = e^{-\frac{k^2}{2}t}$

$$= (2\pi)^{-\frac{d}{2}} \left((\mathcal{F}^{-1} W_t) * f_0 \right)(x)$$

with heat kernel $G(t,x) = (2\pi)^{-\frac{d}{2}} (\mathcal{F}^{-1} W)(t,x) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{x^2}{2t}}$ we find

$$f(t,x) = \left(G(t) * f_0\right)(x) = (2\pi t)^{-\frac{d}{2}} \int e^{-\frac{(x-y)^2}{2t}} f_0(y)\,dy$$

To summarize: The solution to the free SE is

$\psi_0(t) = \mathcal{F}^{-1} M_{P_f} \mathcal{F} \psi_0 = e^{-i(\frac{-\Delta}{2})t} \psi_0 = G(t) * \psi_0$, with $P_f(k) = e^{-i\frac{k^2}{2}t}$, $G(t,x) = (2\pi i t)^{-\frac{d}{2}} e^{\frac{i(x-y)^2}{2t}}$

2.2 Tempered Distributions

Definition 2.27:

Let V be a topological vector space over a field F (here usually $F=\mathbb{C}$).

Then the dual space V' is the space of all continuous linear maps $V\to F$.

For $f\in V$, $T\in V'$ we write $\underbrace{T(f)}_{\in F}=(f,T)_{V,V'}$ "natural pairing"

Note: • In finite dimensional vector spaces, elements in V (in some basis, column vector) can be identified with elements in V' (row vector)

• But in infinite dimensional spaces, V' is "larger" than V (dual to basis in V is not necessarily a basis)

Definition 2.26:

The elements of the dual space $S'(\mathbb{R}^d)$ of $S(\mathbb{R}^d)$ are called tempered distributions (or "generalized functions").

Examples:

• Let $(1+|x|^2)^{-m}g(x)\in L^1(\mathbb{R}^d)$ for some $m\in\mathbb{N}$; define

$$T_g: S(\mathbb{R}^d)\to\mathbb{C},\ f\longmapsto\int g(x)f(x)\,dx$$

↳ T_g linear clear

↳ T_g continuous? If $f_n\xrightarrow{n\to\infty}f$ in S, does $T_g(f_n-f)\to 0$ (as a sequence in $\mathbb{C}$)?

$$|T_g(f_n - f)| = \left| \int g(x) \big(f_n(x) - f(x)\big)\, dx \right| \leq \int |g(x)|\, |f_n(x) - f(x)|\, dx$$

$$\leq \underbrace{\int (1+|x|^2)^{-m} |g(x)|\, dx}_{< \infty} \; \underbrace{\| (1+|x|^2)^m |f_n(x) - f(x)| \|_\infty}_{\xrightarrow{n\to\infty} 0}$$

$$\Rightarrow T_g \in \mathcal{S}'$$

- Delta distribution $\delta : \mathcal{S} \to \mathbb{C},\ f \mapsto \delta(f) = f(0)$

$\Rightarrow \delta \in \mathcal{S}'$ clear $\left(|f_n(0) - f(0)| \leq \|f_n - f\|_\infty \right)$

A useful notation (in the spirit of previous example) is

$\delta(f) = f(0) = \int \delta(x) f(x)\, dx$, and similarly $\int \delta(x-a) f(x)\, dx = f(a)$

but keep in mind that $\delta(x)$ is not a function $\mathbb{R}^d \to \mathbb{C}$!