Last time we discussed the delta distribution $\mathcal{S}' \ni \delta : \mathcal{S} \to \mathbb{C}, f \mapsto f(0)$

It is a generalized function that can be approximated by functions, e.g., in the following way:

Let $g \in L^1(\mathbb{R})$ ($d=1$ here), $\int g(x)dx = 1$ and $g_n(x) = n\, g(nx)$ (a dilation as in HW 2)

s.t. $\int g_n(x)dx = \int n\, g(nx)\, dx = \int g(y)dy = 1$

$$\Rightarrow \lim_{n\to\infty} T_{g_n}(f) = \lim_{n\to\infty} \int g_n(x) \underbrace{f(x)}_{= f(0) + f(x) - f(0)} dx$$

$$= f(0) + \lim_{n\to\infty} \underbrace{\int n\, g(nx) \big(f(x) - f(0)\big) dx}_{= \int g(y) \big(\underbrace{f(\frac{y}{n}) - f(0)}_{\xrightarrow{n\to\infty} 0 \text{ pointwise}} \big) dy \ \xrightarrow{n\to\infty} 0 \text{ by dominated convergence}}$$

$$= f(0) = \delta(f)$$

Next: We have two natural notions of convergence (for (f_n), $f_n \in V$, and $(T_n)_n$, $T_n \in V'$).

Definition 2.29: Let V be a topological vector space. We define:

a) $(f_n)_n$, $f_n \in V$ $\boxed{\text{converges weakly}}$ to $f \in V$ if $\lim_{n\to\infty} T(f_n) = T(f)$ $\forall\, T \in V'$.

We use the notation: $\text{w-}\lim_{n\to\infty} f_n = f$ or $f_n \rightharpoonup f$

b) $(T_n)_n$, $T_n \in V'$ is a weak* convergent sequence with limit $T \in V'$ if

$$\lim_{n\to\infty} T_n(f) = T(f) \quad \forall f \in V$$

We use the notation: $\text{w}^*\text{-}\lim\limits_{n\to\infty} T_n = T$ or $T_n \xrightarrow{*} T$

Ex.: $T_{g_n} \xrightarrow{*} \delta$

Next: extend $\mathcal{F}$ and ∂_x^α to operators $S' \to S'$

Theorem 2.30:

Let $A: S \to S$ be linear and continuous. Then the adjoint $A': S' \to S'$, defined via

$$\overbrace{(\underbrace{\underbrace{A'T}_{\in S'}}_{\in S'})\ (\underbrace{f}_{\in S})}^{\in \mathbb{C}} := \underbrace{T(Af)}_{=(f,A'T)_{S,S'}=(Af,T)_{S,S'}} \quad \forall f \in S,$$ is a weak* continuous linear map.

Proof: First, $A'T \in S'$, since $T \circ A$ composition of continuous maps.

Sequential continuity: Let $T_n \xrightarrow{*} T$, then $\forall f \in S$:

$$(A'T_n)(f) := T_n(Af) \xrightarrow{n\to\infty} T(Af) = (A'T)(f), \text{ so } A'T_n \xrightarrow{*} AT \ \checkmark$$

Problem: topology in S' not given by a metric, so sequential continuity does not necessarily imply continuity

But here it does, using the topological concept of nets (proof omitted). $\square$

Definition 2.31: $\mathcal{F}_{S'} := \mathcal{F}_S'$, meaning for $T \in S'$, we define its Fourier transform

$$\hat{T} \in S' \text{ by } \hat{T}(f) = T(\hat{f}) \quad \forall f \in S.$$

Corollary 2.32: $\mathcal{F}': \mathcal{S}' \to \mathcal{S}'$ is a weak*-continuous bijection, and $\underbrace{\hat{T}_f = T_{\hat{f}}}$ for all $f \in \mathcal{S}$ (or $f \in L^1$) (recall $T_f(g) := \int fg$).

i.e., $\hat{T}_f(g) = T_{\hat{f}}(g) := \int \hat{f} g \ \forall g \in \mathcal{S}$

Proof: $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ is continuous and linear, so we conclude with Thm. 2.30 that $\mathcal{F}': \mathcal{S}' \to \mathcal{S}'$ is weak*-continuous.

Bijective? $(\mathcal{F}^{-1\prime}\mathcal{F}'T)(f) = (\mathcal{F}'T)(\mathcal{F}^{-1}f) = T(\mathcal{F}\mathcal{F}^{-1}f) = T(f)$

$\Rightarrow$ yes, with continuous inverse $\mathcal{F}'^{-1} = \mathcal{F}^{-1\prime}$.

Also, for $f \in \mathcal{S}$ or $f \in L^1$:

$$\hat{T}_f(g) = (\mathcal{F}T_f)(g) = T_f(\mathcal{F}g) = \int f(x)\hat{g}(x)dx \overset{\text{Plancherel}}{=} \int \hat{f}(x)g(x) = T_{\hat{f}}(g) \quad \forall g \in \mathcal{S} \qquad \square$$

Ex.: Fourier transform of δ ($\delta(f) = f(0)$)

$$\Rightarrow \hat{\delta}(f) = \delta(\hat{f}) = \hat{f}(0) = \int \underbrace{(2\pi)^{-\frac{d}{2}}}_{g(x)} f(x)\,dx = T_g(f)$$

$\Rightarrow T_g$ with $g(x) = (2\pi)^{-\frac{d}{2}}$ is the Fourier transform of δ, or "$\hat{\delta}(k) = (2\pi)^{-\frac{d}{2}}$"

Next: derivatives

Note: $\partial_x^\alpha: \mathcal{S} \to \mathcal{S}$ is linear (clear) and continuous, since

$$\|\partial_x^\alpha f\|_{\gamma,\beta} = \|x^\gamma \partial_x^\beta \partial_x^\alpha f\|_\infty = \|f\|_{\gamma,\alpha+\beta}$$ (i.e., continuity on $\mathcal{S}$ follows as usual from sequential continuity)

Definition 2.34: $\tilde{\partial}_x^\alpha := \left((-1)^{|\alpha|}\partial_x^\alpha\right)' : \mathcal{S}' \to \mathcal{S}'$, i.e., for $T \in \mathcal{S}'$ the distributional derivative $\tilde{\partial}_x^\alpha T$ is defined by $(\tilde{\partial}_x^\alpha T)(f) := T\left((-1)^{|\alpha|}\partial_x^\alpha f\right)$ $\forall f \in \mathcal{S}$.

Corollary 2.35: $\tilde{\partial}_x^\alpha : \mathcal{S}' \to \mathcal{S}'$ is weak*-continuous and $\tilde{\partial}_x^\alpha T_g = T_{\partial_x^\alpha g}$ $\forall g \in \mathcal{S}$.

Proof: Weak*-continuity follows again from Thm. 2.30.

Also, $$\left(\tilde{\partial}_x^\alpha T_g\right)(f) = T_g\left((-1)^{|\alpha|}\partial_x^\alpha f\right) = \int g(x)\,(-1)^{|\alpha|}\partial_x^\alpha f(x)\,dx$$

$|\alpha|$ times integration by parts

$$= \int \left(\partial_x^\alpha g(x)\right) f(x)\,dx = T_{\partial_x^\alpha g}(f) \quad \forall f \in \mathcal{S}.$$

Ex.:
- For $\Theta(x) = \mathbb{1}_{[0,\infty)}(x) := \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$, we find $\frac{d}{dx}\Theta = \delta$, see HW.
- $\tilde{\partial}_x^\alpha \delta$? See HW.