Last time we have defined $\tilde{\mathcal{F}} = \mathcal{F}' : S' \to S'$ and $\tilde{\partial}_x^\alpha = (-1)^{|\alpha|} \partial_x^{\alpha\,\prime} : S' \to S'$.

Furthermore one can show:

- Fixing $h \in S$, we can define the convolution $h \tilde{*} \cdot : S' \to S'$ via $(h \tilde{*} T)(f) = T(\tilde{h} * f)$ with $\tilde{h}(x) = h(-x)$. This definition is chosen such that $\underbrace{h \tilde{*} T_g = T_{g*h}}$ for $g \in S$.

  $(h\tilde{*}T_g)(f) := T_g(\tilde{h} * f) := \int dx\, g(x) \int dy\, h(y-x) f(y)$
  $= \int dy\, f(y) \int dx\, h(y-x) g(x)$

- Fixing $g \in C^\infty_{pol}$, we define $\tilde{M}_g = M_g'$, i.e., $(M_g T)(f) = T(\underbrace{M_g f}_{=gf})$.

  Note: gT well-defined for $g \in C^\infty_{pol}$, but product of distributions a-priori undefined (much research effort to define it at least for some distributions, e.g., Hairer's regularity structures).

Both are weak* continuous maps.

Note: $\{T_f \in S' : f \in S\}$ is dense in S' w.r.t. weak*-topology (not obvious, proof omitted).
Thus, T_f allows us to identify S with some subset of S'.
Because of density and continuity of the adjoint, the definition $A' T_f = T_{Af}$ uniquely defines A' on all of S'. ← This is why we defined, e.g., $\tilde{\partial}_x^\alpha = (-1)^{|\alpha|} \partial_x^{\alpha\,\prime}$.

From now on, we will forget about $\sim$ or $'$ in the notation for the adjoint.

$\Rightarrow$ We have defined $\mathcal{F}T$, $\partial_x^\alpha T$, $h*T$ for $h \in S$, gT for $g \in C^\infty_{pol}$ $(T \in S')$.

With that we can solve the free Schrödinger equation on $\mathcal{S}'$:

Theorem 2.40:

Let $\psi_0 \in \mathcal{S}'$, then the unique global solution to the free Schrödinger equation

$i\partial_t \psi = -\frac{1}{2}\Delta\psi$ (in the sense of distributions) with $\psi(0) = \psi_0$ is $\psi(t) = \mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F}\psi_0$,

with $\psi \in C^\infty(\mathbb{R}_t, \mathcal{S}'(\mathbb{R}^d))$.

Proof: First, note that $\psi(t) = \mathcal{F}^{-1} e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0 \in \mathcal{S}'$ since $\mathcal{F}, \mathcal{F}^{-1}, M_f : \mathcal{S}' \to \mathcal{S}'$.

Next, let us check if this $\psi(t)$ solves the SE. For any $f \in \mathcal{S}$, we find

$$
\begin{aligned}
i\frac{d}{dt}(f, \psi(t))_{\mathcal{S},\mathcal{S}'} &= i\frac{d}{dt}\left(f, \mathcal{F}^{-1} e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0\right)_{\mathcal{S},\mathcal{S}'} \\
&\overset{\text{by def.}}{=} i\frac{d}{dt}\left(\mathcal{F} e^{-i\frac{k^2}{2}t}\mathcal{F}^{-1} f, \psi_0\right)_{\mathcal{S},\mathcal{S}'} \\
&\overset{\substack{\text{continuity} \\ \text{of } \psi_0 : \mathcal{S} \to \mathbb{C}}}{=} \left(\mathcal{F}\left(\frac{id}{dt} e^{-i\frac{k^2}{2}t}\right)\mathcal{F}^{-1} f, \psi_0\right)_{\mathcal{S},\mathcal{S}'} \\
&= \left(\mathcal{F} e^{-i\frac{k^2}{2}t} \underbrace{\frac{k^2}{2}\mathcal{F}^{-1} f}_{=\mathcal{F}^{-1}(-\frac{\Delta}{2}f)}, \psi_0\right)_{\mathcal{S},\mathcal{S}'} \\
&= \left(-\frac{\Delta}{2} f, \mathcal{F}^{-1} e^{-i\frac{k^2}{2}}\mathcal{F}\psi_0\right)_{\mathcal{S},\mathcal{S}'} \\
&\overset{\substack{\text{by def. of the} \\ \text{distributional} \\ \text{derivative}}}{=} \left(f, -\frac{\Delta}{2}\psi(t)\right)_{\mathcal{S},\mathcal{S}'}.
\end{aligned}
$$

Similarly $\left(i\frac{d}{dt}\right)^k (f, \psi(t))_{\mathcal{S},\mathcal{S}'} = \left(\left(-\frac{\Delta}{2}\right)^k f, \psi(t)\right)_{\mathcal{S},\mathcal{S}'}$, so $\psi(t) \in C^\infty(\mathbb{R}_t, \mathcal{S}'(\mathbb{R}^d))$. $\square$

Summary on $i\partial_t \psi(t,x) = -\frac{\Delta}{2}\psi(t,x)$:

- First approach: regard this as a (classical) PDE for (classically) differentiable functions $\psi(t,x)$.
- Second approach: regard this as an ODE $i\frac{d}{dt}\psi(t) = -\frac{\Delta}{2}\psi(t)$ for $\psi(t) \in \mathcal{S}$ (or more generally some function space), i.e., $\psi: \mathbb{R} \to \mathcal{S}$. Then show that $\psi \in C^1(\mathbb{R}, \mathcal{S})$, or even $C^\infty(\mathbb{R}, \mathcal{S})$ as in our case (or some other space as appropriate).
- Third approach: regard this as an ODE in the distributional sense, i.e., $\psi: \mathbb{R} \to \mathcal{S}'$ ($-\frac{\Delta}{2}$ is then not the classical Laplacian, but the distributional Laplacian). Then show, e.g., $\psi \in C^p(\mathbb{R}, \mathcal{S}')$ for some $1 \leq p \leq \infty$.

2.3 Long-time Asymptotics and the Momentum Operator

Let us consider the solution $\psi(t) \in \mathcal{S}$ of the free SE.

Recall: probability that particle at time t is in $\Lambda \subset \mathbb{R}^d$ is $\mathbb{P}(X(t) \in \Lambda) = \int_\Lambda |\psi(t,x)|^2 dx$.

What about momentum (=velocity here, since mass $m=1$)? A-priori not defined in QM.

Let us consider the asymptotic velocity $= \frac{\text{distance}}{\text{time}}$ for large times t.

Probability that velocity is in $\Gamma \subset \mathbb{R}^d$ is $\mathbb{P}(\frac{X(t)}{t} \in \Gamma) = \mathbb{P}(X(t) \in t\Gamma) = \int_{t\Gamma} |\psi(t,x)|^2 dx$.

So next, we try to find $\lim_{t\to\infty} \mathbb{P}(\frac{X(t)}{t} \in \Gamma)$.

Let's compute:

$$\psi(t,x) := (2\pi i t)^{-\frac{d}{2}} \int e^{i\frac{(x-y)^2}{2t}} \psi_0(y)\,dy$$

$$= \frac{e^{i\frac{x^2}{2t}}}{(it)^{d/2}} (2\pi)^{-\frac{d}{2}} \int e^{-i\frac{x}{t}y} \left(e^{i\frac{y^2}{2t}} - 1 + 1\right) \psi_0(y)\,dy$$

$$= \frac{e^{i\frac{x^2}{2t}}}{(it)^{d/2}} \hat{\psi}_0\left(\frac{x}{t}\right) + \underbrace{\frac{e^{i\frac{x^2}{2t}}}{(it)^{d/2}} \hat{h}_t\left(\frac{x}{t}\right)}_{=: r(t,x)} \quad \text{, where } h_t(y) = \left(\underbrace{e^{i\frac{y^2}{2t}} - 1}\right) \psi_0(y).$$

← should be small for large t, since goes to 0 as $t \to \infty$