Foundations of Mathematical Physics, Jacobs University, Fall 2021 — Session 13

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We continue our study of the asymptotic velocity.

Last time we established that the probability density $\rho_\psi(t,x) = |\psi(t,x)|^2$ and the probability current $j_\psi(t,x) = \operatorname{Im} \overline{\psi(t,x)} (\nabla \psi(t,x))$ satisfy the continuity equation

$$\boxed{\partial_t \rho_\psi + \nabla j_\psi = 0}$$ (also with potentials, i.e., when $H = -\Delta + V$, V real).

Note: The continuity eq. implies: $\underbrace{\partial_t \int_\Lambda \rho_\psi dx}_{\text{change of mass/probability/... in } \Lambda \subset \mathbb{R}^d \text{ (compact)}} = -\int_\Lambda \nabla j_\psi dx \overset{\text{Gauss (Stokes)}}{=} \underbrace{-\int_{\partial\Lambda} j_\psi d\sigma}_{\text{flow through boundary of } \Lambda}$

Now: current = density · velocity, i.e., $j_\psi = \rho_\psi \cdot v_\psi$

$\Rightarrow$ velocity vector field $v_\psi(t,x) = \frac{j_\psi(t,x)}{\rho_\psi(t,x)} = \frac{\operatorname{Im} \overline{\psi(t,x)} \nabla \psi(t,x)}{\overline{\psi(t,x)} \psi(t,x)} = \operatorname{Im} \underbrace{\frac{\nabla \psi(t,x)}{\psi(t,x)}}$

looks dangerous at zeros of ψ, but since $\rho_\psi = 0$ at zeros of ψ, the velocity field never needs to be evaluated at the zeros (particles are never there).

Let us approximate v_ψ for large t, or v_{ψ_ε} for $\varepsilon \to 0$

$\hookrightarrow$ recall: macroscopic wave fct. $\psi_\varepsilon(t,x) = \varepsilon^{-\frac{d}{2}} \psi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$,

i.e., $\psi_\varepsilon(t,x) = \frac{e^{i\frac{x^2}{2\varepsilon t}}}{(it)^{d/2}} \hat{\psi}_0\left(\frac{x}{t}\right) + \underbrace{\varepsilon^{-\frac{d}{2}} r\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)}_{\to 0 \text{ in } L^2 \text{ as } \varepsilon \to 0}$

On the macro scale, we have

$$v_\psi\left(\tfrac{t}{\varepsilon},\tfrac{x}{\varepsilon}\right) = \operatorname{Im}\frac{(\nabla\psi)\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)}{\psi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)} = \varepsilon \operatorname{Im}\frac{\nabla_x \psi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)}{\psi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)} = \varepsilon\, v_{\psi_\varepsilon}(t,x)$$

We do not discuss this step rigorously

$$\approx \varepsilon \operatorname{Im}\frac{\nabla_x\left(e^{i\frac{x^2}{2\varepsilon t}}\hat{\psi}_0\left(\frac{x}{t}\right)\right)}{e^{i\frac{x^2}{2\varepsilon t}}\hat{\psi}_0\left(\frac{x}{t}\right)}$$

$$= \varepsilon \operatorname{Im} i\frac{x}{\varepsilon t} + \mathcal{O}(\varepsilon)$$

$$= \frac{x}{t} + \mathcal{O}(\varepsilon)$$

$\Rightarrow v_\psi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right) \approx \frac{x}{t}$ for small ε (or $v_\psi(t,x) \approx \frac{x}{t}$ for large t)

↳ in this sense classical trajectories appear in QM

A small digression: We can regard $v_\psi(t,x)$ as an actual velocity vector field for particles. This indeed yields a coherent mathematical and physical theory of quantum mechanics, called Bohmian Mechanics (or de Broglie-Bohm theory).

We have two defining equations:

Schrödinger eq.: $\boxed{i\partial_t \Psi(t) = H\Psi(t)}$ for $\Psi \in L^2(\mathbb{R}^{dN})$, with initial condition $\Psi(0) = \Psi_0$

Law of Motion for particles: $\boxed{\frac{d}{dt}X_i(t) = \operatorname{Im}\frac{\nabla_i\Psi}{\Psi}\left(t, x = (X_1(t),\dots,X_N(t))\right)}$ for $X_i \in \mathbb{R}^d$, $i = 1,\dots,N$, with initial condition $X_i(0) = x_{i,0}$.

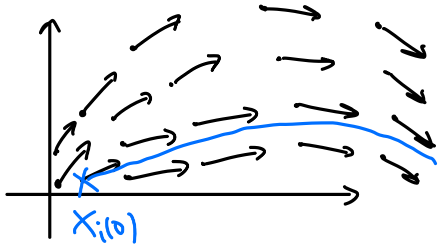
many-particle generalization of velocity vector field

Here we see non-locality again: Trajectory of particle i depends on all other trajectories!

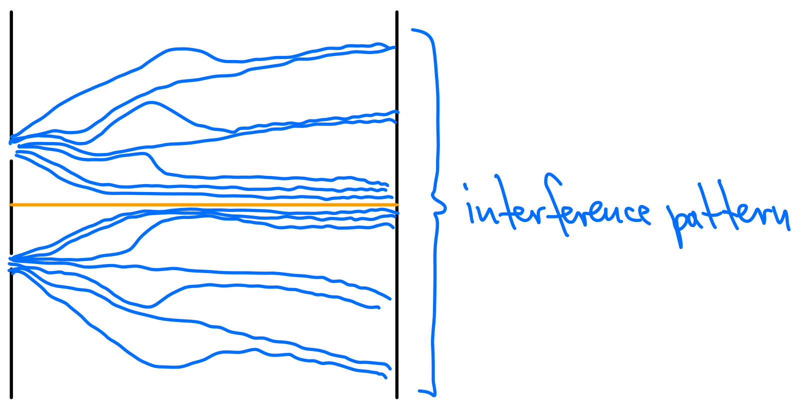
$\Rightarrow$ The particles are "guided by the wave function".

How do the trajectories look like?

They are integral curves of the vector field v_Ψ



- Ex.: We have seen above that free particles asymptotically move on straight lines with constant velocity (just as classical particles). But this is not true for small times or in the presence of interaction.
- Ex.: If Ψ is a ground state, one can prove that it can always be chosen real. In that case $\frac{d}{dt} X_i(t) = 0$, i.e., $X_i(t) = X_i(0)$: nothing moves.
- Ex.: Double slit with reasonable initial data:



- Since law of motion is first order, particle trajectories can never cross.

The particle trajectories are so important because they allow us to analyze subsystems.

E.g., we can ask questions such as:

a) Assuming in a large system particles are $|\Psi|^2$ distributed. Does this imply that particles in subsystems are also $|\Psi_{sub}|^2$ distributed? Under which conditions, in which sense?

b) Assuming we seperate our system into "measurement apparatus" and "system to be measured", modeling the apparatus in a reasonable way. Does this imply a "collapse" of the effective system wave function?

The key to answering such question is the concept of conditional wave function.
Let $x_1,\dots,x_N$ be the variables of "system 1" (e.g., a small subsystem), and $Y_1,\dots,Y_M$ the variables of "system 2" (e.g., a larger "rest", or measurement apparatus). Then

$$\boxed{\Psi_{cond}(x_1,\dots,x_N) := \mathcal{N}\,\Psi(x_1,\dots,x_N,\underbrace{Y_1(t),\dots,Y_M(t)}_{\text{we plug in the actual configuration for system 2}})}\,,$$

where $\mathcal{N} = \left[\int |\Psi(x_1,\dots,x_N,Y_1(t),\dots,Y_M(t))|^2\,dx_1\dots dx_N\right]^{-\frac{1}{2}}$ is a normalization constant.

One can then show that the questions above can be answered as:

a) The conditional wave function satifies its own subsystem Schrödinger eq., and particles in the subsystem are $|\Psi_{cond}|^2$ distributed, if there is no entanglement and no interaction between subsystem and rest.

b) Yes, the conditional wave function indeed "collapses" (while the full Ψ never does!)