A few consequences of ONBs:

- Proposition 3.10: $\boxed{\text{An ONS } (\varphi_j)_j \text{ is an ONB iff: } \langle\varphi_j,\psi\rangle = 0 \ \forall j\in\mathbb{N} \Rightarrow \psi = 0}$ (proof: HW)

- Parseval's identity: $\boxed{\|\psi\|^2 = \sum_{j=1}^{\infty} |\langle\varphi_j,\psi\rangle|^2}$ for $(\varphi_j)_j$ an ONB

Proof: $\|\psi\|^2 = \langle \lim_{N\to\infty} \sum_{j=1}^{N} \langle\varphi_j,\psi\rangle\varphi_j , \lim_{M\to\infty} \sum_{i=1}^{M} \langle\varphi_i,\psi\rangle\varphi_i \rangle$

scalar product continuous $\overset{}{=} \lim_{N\to\infty}\lim_{M\to\infty} \underbrace{\langle \sum_{j=1}^{N} \langle\varphi_j,\psi\rangle\varphi_j , \sum_{i=1}^{M} \langle\varphi_i,\psi\rangle\varphi_i \rangle}$

$\{\varphi_j\}$ ONB $= \sum_{j=1}^{N} \overline{\langle\varphi_j,\psi\rangle}\langle\varphi_j,\psi\rangle$

$= \lim_{N\to\infty} \sum_{j=1}^{N} |\langle\varphi_j,\psi\rangle|^2$ □

- $\boxed{U: \mathcal{H}\to \ell^2,\ \psi \mapsto (\langle\varphi_j,\psi\rangle)_{j\in\mathbb{N}} \text{ is an } \overbrace{\text{isometric}}^{\|\psi\|_{\mathcal{H}} = \|(\langle\varphi_j,\psi\rangle)_j\|_{\ell^2}} \ \overbrace{\text{isomorphism}}^{\text{linear bijective map}}}$ (for separable $\mathcal{H}$) otherwise there would not be an ONB

Proof: Isometry due to Parseval $\|\psi\|_{\mathcal{H}}^2 = \sum_{j=1}^{\infty} |\langle\varphi_j,\psi\rangle|^2 = \|(\langle\varphi_j,\psi\rangle)_j\|_{\ell^2}^2 = \|U\psi\|_{\ell^2}^2$

injective clear bc. of isometry

$\underbrace{\text{Isomorphism since } U \text{ surjective}}$: for any $(c_j)_{j\in\mathbb{N}} \in \ell^2$, we choose $\psi = \sum_{j=1}^{\infty} c_j\varphi_j$.

Then $U\psi = (\langle\varphi_j, \sum_{k=1}^{\infty} c_k\varphi_k\rangle)_j = (c_j)_j$ ✓

(and $\psi\in\mathcal{H}$ since $\|\sum_{j=N}^{\infty} c_j\varphi_j\|_{\mathcal{H}}^2 = \langle \sum_{j=N}^{\infty} c_j\varphi_j , \sum_{i=N}^{\infty} c_i\varphi_i\rangle = \sum_{j=N}^{\infty} |c_j|^2 \xrightarrow{N\to\infty} 0$ $(c\in\ell^2)$). □

$\Rightarrow$ All infinite dimensional separable Hilbert spaces are isometrically isomorphic to ℓ^2, and thus to each other. (Finite dimensional Hilbert spaces are isometrically isomorphic to $\mathbb{C}^n$.)

↳ ℓ^2 is the coordinate space for any separable Hilbert space (of infinite dimension)
(any choice of ONB gives us an isometric isomorphism)

Example: $(\varphi_k)_{k\in\mathbb{Z}}$ with $\varphi_k = (2\pi)^{-\frac{1}{2}} e^{ikx}$ is an ONB for $L^2([0,2\pi])$;

$$\psi = \sum_{k\in\mathbb{Z}} \langle \varphi_k, \psi\rangle \varphi_k \quad \text{is the Fourier series of } \psi$$

Note: So why are we even interested in different Hilbert spaces? Because we are often interested in extra structure, e.g., operators on Hilbert spaces. Think of Fourier space, where differential operators become multiplication operators. (Or think of diagonalization in $\mathbb{C}^n$.)

Finally, we note that Hilbert spaces can be decomposed orthogonally.

Definition 3.14: For any $\overbrace{M \subset \mathcal{H}}^{\text{any subset}}$, we call $M^\perp := \{\psi \in \mathcal{H} : \langle \varphi, \psi\rangle = 0 \ \forall \varphi \in M\}$ the orthogonal complement of M.

Note: • $M \cap M^\perp = \begin{cases} \{0\} & \text{if } 0 \in M \\ \emptyset & \text{if } 0 \notin M \end{cases}$

• $M^\perp$ is a $\underbrace{\text{closed}}_{\langle\varphi,\cdot\rangle \text{ continuous}}$ $\underbrace{\text{subspace}}_{\langle\varphi,\cdot\rangle \text{ linear}}$ of $\mathcal{H}$

$\boxed{\text{Theorem 3.15: Let } \mathcal{M} \subset \mathcal{H} \text{ be a closed subspace. Then } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp \text{, meaning } \forall \psi \in \mathcal{H} \text{ we have } \psi = \varphi + \varphi^\perp \text{ with unique } \varphi \in \mathcal{M}, \varphi^\perp \in \mathcal{M}^\perp.}$

Proof: We give the proof only for separable $\mathcal{H}$.

$\Rightarrow$ also $\mathcal{M}, \mathcal{M}^\perp$ are separable Hilbert spaces with ONBs $(\varphi_i)_i$, $(\varphi_j^\perp)_j$.

Choose any $\psi \in \mathcal{H}$, def. $\varphi = \sum_{i=1}^\infty \langle \varphi_i, \psi \rangle \varphi_i$, $\varphi^\perp = \sum_{j=1}^\infty \langle \varphi_j^\perp, \psi \rangle \varphi_j^\perp$, then $\psi = \varphi + \varphi^\perp$ if $(\varphi_i)_i \cup (\varphi_j^\perp)_j$ is an ONB of $\mathcal{H}$.

Check with Proposition 3.10: Let $\underbrace{\langle \varphi_i, \psi \rangle = 0}_{} = \langle \varphi_j^\perp, \psi \rangle \quad \forall i, j$

$$\Rightarrow \langle \varphi, \psi \rangle = 0 \ \forall \varphi \in \mathcal{M} \Rightarrow \psi \in \mathcal{M}^\perp$$

but since also $\langle \varphi^\perp, \psi \rangle = 0 \ \forall \varphi^\perp \in \mathcal{M}^\perp \Rightarrow \psi = 0$

Uniqueness: Suppose there is another decomposition $\psi = \underbrace{\tilde{\varphi}}_{\in \mathcal{M}} + \underbrace{\tilde{\varphi}^\perp}_{\in \mathcal{M}^\perp}$

$\Rightarrow \varphi + \varphi^\perp = \tilde{\varphi} + \tilde{\varphi}^\perp$, i.e., $\mathcal{M} \ni \varphi - \tilde{\varphi} = \tilde{\varphi}^\perp - \varphi^\perp \in \mathcal{M}^\perp$

but $\mathcal{M} \cap \mathcal{M}^\perp = \{0\}$ so $\varphi = \tilde{\varphi}$ and $\varphi^\perp = \tilde{\varphi}^\perp$. $\square$

Next topic: operators between normed spaces

In ∞-dimensions, there are not just bounded, but also unbounded operators.
Bounded ones are nicer, so let us consider those first.
(Later: Hamiltonians H will be unbounded, but e^{-iHt} will be bounded.)

Definition 3.16: Let X and Y be normed spaces. A linear operator $L: X \to Y$ is bounded if $\exists\, C < \infty$ with $\underbrace{\|Lx\|_Y}_{\text{norm on } Y} \leq C \underbrace{\|x\|_X}_{\text{norm on } X}$ $\forall x \in X$.

Example: Multiplication operators $M_V: L^p \to L^p$ for $V \in L^\infty$ are bounded (HW 3 Problem 2).

Proposition 3.17: The space $\mathcal{L}(X,Y) = \{L: X \to Y,\ L \text{ linear and bounded}\}$ with norm $\boxed{\|L\|_{\mathcal{L}(X,Y)} := \sup_{\substack{x \in X \\ \|x\|_X = 1}} \|Lx\|_Y}$ is itself a normed space.

If $\underbrace{Y \text{ is a Banach space}}_{\text{not necessarily } X}$, also $\mathcal{L}(X,Y)$ is a Banach space.

Proof: HW 5