Recall: A linear operator $L: \underbrace{X \to Y}_{X, Y \text{ normed spaces}}$ is bounded if its operator norm

$$\|L\| := \sup_{\substack{x \in X \\ \|x\|=1}} \|Lx\| \text{ is finite.}$$

Why are bounded operators so interesting? Because these are also the continuous ones! (And since we deal with linear ones, it is enough to check continuity at 0.)

$$\boxed{\begin{array}{l}\underline{\text{Lemma 3.18:}} \text{ Let } L: X \to Y \text{ be linear } (X, Y \text{ normed spaces}). \text{ Then the following statements} \\ \text{are equivalent: (i) } L \text{ is continuous at } 0. \\ \qquad\qquad\qquad\quad \text{(ii) } L \text{ is continuous.} \\ \qquad\qquad\qquad\quad \text{(iii) } L \text{ is bounded.}\end{array}}$$

<u>Proof:</u> (iii) $\Rightarrow$ (i): Let $\|x_n\|_X \to 0 \Rightarrow \|Lx_n\|_Y \underset{L \text{ bounded}}{\leq} \|L\|_{\mathcal{B}} \|x_n\|_X \to 0$

(i) $\Rightarrow$ (ii): Let $\|x_n - x\|_X \to 0 \Rightarrow \|Lx_n - Lx\|_Y \overset{\text{linearity}}{=} \|L(x_n - x)\|_Y \xrightarrow{\text{continuity at } 0} 0$

(ii) $\Rightarrow$ (iii): suppose L not bounded, then $\exists$ a sequence $(x_n)_n$ with $\|x_n\|_X = 1$ $\forall n \in \mathbb{N}$ and $\underbrace{\|Lx_n\|_Y \geq c(n)}$ for some $c(n) \xrightarrow{n \to \infty} \infty$. Defining $z_n := \frac{x_n}{\|Lx_n\|_Y}$, we have $\|z_n\| = \frac{\|x_n\|_X}{\|Lx_n\|_Y} \leq \frac{1}{c(n)}$, i.e., $z_n \xrightarrow{n \to \infty} 0$. But $\|Lz_n\|_Y = \frac{\|Lx_n\|_Y}{\|Lx_n\|_Y} = 1$, which contradicts continuity (at 0). $\square$

Note: by using subsequences (rescaling the index) we could even use $c(n) = n$.

What do unbounded operators look like? Much more later, here just one example:

Define $\ell_0 = \{ (x_n)_n \in \ell^1 : \exists N \in \mathbb{N} \text{ s.t. } x_n = 0 \ \forall n \geq N \}$ with the norm

$$\|(x_n)_n\|_{\ell^1} = \underbrace{\sum_{n=1}^{\infty}}_{\text{actually just a finite sum}} |x_n| .$$ Define $T: \ell_0 \to \ell_0, x \mapsto Tx = (x_1, 2x_2, 3x_3, \dots)$.

But if $(e_k^{(n)})_k$ is the sequence with $e_k^{(n)} = \begin{cases} 1 & \text{for } k=n \\ 0 & \text{otherwise} \end{cases}$, in particular $\|e^{(n)}\| = 1$,

then $\|Te^{(n)}\| = n$, i.e., T is unbounded.

In the last chapter, we defined operators on S' by defining them on a dense subset and extending them by continuity (but we did not fully prove this). This can also be done here (for bounded = continuous operators):

Theorem 3.20: Let Z be a dense subspace of a normed space X, and let Y be a Banach space. Let $L: Z \to Y$ be a linear bounded operator. Then L has a unique linear bounded extension $\tilde{L}: X \to Y$ with $\underbrace{\tilde{L}|_Z = L}_{\tilde{L} \text{ and } L \text{ coincide on } Z}$ and $\|\tilde{L}\|_{\mathcal{L}(X,Y)} = \|L\|_{\mathcal{L}(Z,Y)}$.

Proof: Idea: using continuity we "fill in the gaps."

Choose some $x \in X$, then $\exists$ sequence $(z_n)_n$ in Z with $\|z_n - x\|_X \to 0$ (using just density of Z in X; note: $x \in X$ is fixed, no completeness necessary).

$\Rightarrow (z_n)_n$ converges $\Rightarrow (z_n)_n$ is a Cauchy sequence.

$\Rightarrow \|Lz_n - Lz_m\|_Y \overset{\text{linearity}}{=} \|L(z_n - z_m)\|_Y \leq \|L\|_{\mathcal{L}(Z,Y)} \|z_n - z_m\|_Z$, i.e., also $(Lz_n)_n$ is a Cauchy sequence in Y. Since Y is complete, $Lz_n \to y \in Y$.

But is this y independent of the choice of sequence?

Yes: if $\|z_n' - x\|_X \to 0$, also the sequence $(z_1, z_1', z_2, z_2', z_3, z_3', \ldots)$ converges to x and as above $(Lz_1, Lz_1', Lz_2, Lz_2', \ldots)$ converges to some $\tilde{y} \in Y$. But every subsequence of a convergent sequence converges to the same limit.

So we def. $\tilde{L}x := y$ with this construction.

$\|\tilde{L}\|_{\mathcal{L}(X,Y)} \leq \|L\|_{\mathcal{L}(Z,Y)}$

(and $\|L\|_{\mathcal{L}(Z,Y)} \leq \|\tilde{L}\|_{\mathcal{L}(X,Y)}$ clear by def.)

↳ linearity clear

↳ boundedness: $\|\tilde{L}x\|_Y = \lim_{n\to\infty} \underbrace{\|Lz_n\|_Y}_{\leq \|L\|_{\mathcal{L}(Z,Y)} \|z_n\|_Z} \leq \|L\|_{\mathcal{L}(Z,Y)} \|x\|_X \Rightarrow \tilde{L}$ continuous

and continuity on a dense subset implies that this is the unique extension. □

Now, e.g., extension of the Fourier transform from $\mathcal{S}$ to L^2 follows as a simple corollary. Let us first note:

Theorem 3.21: $\underbrace{C_c^\infty(\mathbb{R}^d)}_{\text{smooth functions with compact support}}$ is dense in $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$.

Proof: From HW 3, Problem 3(b), we know that C_c^∞ is dense in C_c $\underbrace{\text{w.r.t. } \|\cdot\|_{L^p}}$.

(We used convolution there to "smoothen out" (or "mollify") $f \in L^p$.)

density is defined w.r.t. a norm, or generally a topology (a subset might be dense w.r.t. one norm, but not another)

It is also a standard result that C_c is dense in L^p, which implies that C_c^∞ is dense in L^p (by a triangle argument). □

Then we have

Theorem 3.22: The Fourier transform $\mathcal{F}: (\mathcal{S}(\mathbb{R}^d), \|\cdot\|_{L^2(\mathbb{R}^d)}) \longrightarrow L^2(\mathbb{R}^d)$ can be uniquely extended to a bounded linear operator $L^2 \to L^2$.

Furthermore:
- $\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2} \quad \forall f \in L^2$
- $\mathcal{F}\mathcal{F}^{-1} = \mathcal{F}^{-1}\mathcal{F} = \mathrm{id}_{L^2}$
- $(\mathcal{F}f)(k) = \underbrace{\lim_{N\to\infty}}_{L^2 \text{ limit, not pointwise}} (2\pi)^{-\frac{d}{2}} \int_{|x|<N} e^{-ikx} f(x)\,dx \quad \forall f \in L^2.$

Proof: $C_c^\infty \subset \mathcal{S} \subset L^2$, so with Thm. 3.21 also $\mathcal{S}$ is dense in L^2 and we can apply Thm. 3.20. (Note: $\mathcal{F}: (\mathcal{S}, \|\cdot\|_{L^2}) \to L^2$ is indeed bounded, since $\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2}$.)

Also: $\mathcal{F}\mathcal{F}^{-1}|_{\mathcal{S}} = \mathcal{F}^{-1}\mathcal{F}|_{\mathcal{S}} = \mathrm{id}_{L^2}|_{\mathcal{S}}$, but since $\mathcal{F}, \mathcal{F}^{-1}$, id continuous, equality holds on L^2.

Limit formula follows directly from $\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2}$: let us denote

$f_N(x) = f(x)\, \underbrace{\mathbb{1}_{B_N(0)}(x)}_{= \begin{cases} 1 & \text{for } |x|<N \\ 0 & \text{else} \end{cases}}$. Then $\lim_{N\to\infty} \|\mathcal{F}f - \mathcal{F}f_N\|_{L^2} = \lim_{N\to\infty} \|f - f_N\|_{L^2} = 0$. $\square$

Note:
- one can of course use any other suitable limit formula for explicit computations.
- so even for functions $\notin L^1$, we have defined $\int f(x) e^{-ikx}\,dx$.