Last time: We defined Fourier transform $\mathcal{F}: L^2 \to L^2$ (and $\mathcal{F}^{-1}: L^2 \to L^2$), using that $\mathcal{F}: (S, \|\cdot\|_{L^2}) \to L^2$ is bounded, i.e., continuous, and that S is dense in L^2.

Note that $\mathcal{F}: L^2 \to L^2$ is a unitary operator:

<u>Definition 3.23:</u> Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. A linear bounded operator $U \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is called unitary if it is surjective and isometric (isometric meaning $\|U\psi\|_{\mathcal{H}_2} = \|\psi\|_{\mathcal{H}_1}$ $\forall \psi \in \mathcal{H}_1$).

Note: • injective follows from $\|U\psi\|_{\mathcal{H}_2} = \|\psi\|_{\mathcal{H}_1}$, so unitary operators are bijective

• with the polarization identity isometry $\Leftrightarrow$ preservation of inner product:

$$\langle U\psi, U\varphi\rangle_{\mathcal{H}_2} = \langle \psi, \varphi\rangle_{\mathcal{H}_1} \quad \forall \psi, \varphi \in \mathcal{H}_1$$

Having $\mathcal{F}: L^2 \to L^2$, we can now solve the free Schrödinger equation on L^2:

For any $t \in \mathbb{R}$, the free propagator on L^2 is $P_f(t): L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $P_f(t) = \mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F}$.

$\Rightarrow P_f(t)$ is clearly unitary ($|e^{-i\frac{k^2}{2}t}| = 1$ and $\mathcal{F}$ isometric) for any $t \in \mathbb{R}$.

To talk about continuity and differentiability of $P_f(t)$, i.e., of $P_f: \mathbb{R} \to \mathcal{L}(L^2)$, we need to distinguish different notions of convergence for bounded operators.

Definition 3.26: Let $(A_n)_n$ be a sequence in $\mathcal{L}(\mathcal{H})$ and $A \in \mathcal{L}(\mathcal{H})$.

a) $(A_n)_n$ converges in norm (or "uniformly") to A if $\lim\limits_{n\to\infty} \|A_n - A\|_{\mathcal{L}(\mathcal{H})} = 0$.

Notation: $\lim\limits_{n\to\infty} A_n = A$, or $A_n \longrightarrow A$.

b) $(A_n)_n$ converges strongly (or "pointwise") to A if $\lim\limits_{n\to\infty} \|A_n\psi - A\psi\|_{\mathcal{H}} = 0 \ \forall \psi \in \mathcal{H}$.

Notation: $\operatorname*{s-lim}\limits_{n\to\infty} A_n = A$, or $A_n \xrightarrow{s} A$.

c) $(A_n)_n$ converges weakly to A if $\lim\limits_{n\to\infty} |\langle \varphi, (A_n - A)\psi\rangle| = 0 \quad \forall \varphi, \psi \in \mathcal{H}$.

Notation: $\operatorname*{w-lim}\limits_{n\to\infty} A_n = A$, or $A_n \xrightarrow{w} A$.

Note: • $|\langle \varphi, (A_n - A)\psi\rangle| \leq \|\varphi\| \, \|(A_n - A)\psi\|_{\mathcal{H}} \leq \|\varphi\| \, \|\psi\| \, \|A_n - A\|_{\mathcal{L}(\mathcal{H})}$,

so norm convergence $\Rightarrow$ strong convergence $\Rightarrow$ weak convergence.

But the other way around is not true; come up with counterexamples in HW 7, Problem 2.

Let us now check continuity and differentiability of $P_f : \mathbb{R} \to \mathcal{L}(L^2)$:

• Uniformly continuous? $\|P_f(t+h) - P_f(t)\|_{\mathcal{L}(L^2)} = \sup\limits_{\substack{\varphi \in L^2 \\ \|\varphi\| = 1}} \|P_f(t+h)\varphi - P_f(t)\varphi\|_{L^2}$

$$= \sup_{\substack{\varphi \in L^2 \\ \|\varphi\| = 1}} \left\| \left(e^{-i\frac{k^2}{2}(t+h)} - e^{-i\frac{k^2}{2}t}\right) \mathcal{F}\varphi \right\|_{L^2}$$

$$= \sup_{\substack{\tilde{\varphi} \in L^2 \\ \|\tilde{\varphi}\| = 1}} \left\| \left(e^{-i\frac{k^2}{2}(t+h)} - e^{-i\frac{k^2}{2}t}\right) \tilde{\varphi} \right\|_{L^2}$$

Problem 2 HW3: $\|M_V\|_{\mathcal{L}(L^2)} = \|V\|_\infty$

$$= \sup_{k\in\mathbb{R}^d} \underbrace{\left| e^{-i\frac{k^2}{2}(t+h)} - e^{-i\frac{k^2}{2}t} \right|}_{= \left|e^{-i\frac{k^2}{2}h}-1\right|}$$

$$= 2 \quad \text{for all } h \neq 0.$$

So $\lim_{h\to 0} \|P_f(t+h) - P_f(t)\|_{\mathcal{L}(L^2)} = 2$, i.e., $P_f(t)$ is not uniformly continuous.

- Strongly continuous? $\|P_f(t+h)\psi_0 - P_f(t)\psi_0\|_{L^2}^2 = \|\psi(t+h) - \psi(t)\|_{L^2}^2$

$$= \left\| \mathcal{F}^{-1}\left(e^{-i\frac{k^2}{2}t} e^{-i\frac{k^2}{2}h} - e^{-i\frac{k^2}{2}t}\right) \overbrace{\mathcal{F}\psi_0}^{\hat{\psi}_0} \right\|^2$$

$$= \int \underbrace{\left| e^{-i\frac{k^2}{2}h} - 1 \right|^2}_{\xrightarrow{h\to 0} 0} |\hat{\psi}_0(k)|^2 \, dk \xrightarrow{h\to 0} 0,$$

by dominated convergence ($\hat{\psi}\in L^2 \Leftrightarrow \psi\in L^2$)

i.e., $P_f(t)$ is strongly continuous on L^2 $\Leftrightarrow$ $\psi(t)$ is continuous $\forall \psi_0 \in L^2$.

- Strongly differentiable? $\left(\frac{\|P_f(t+h)\psi_0 - P_f(t)\psi_0\|_{L^2}}{h}\right)^2 = \left(\frac{\|\psi(t+h)-\psi(t)\|_{L^2}}{h}\right)^2$

$$= \int \underbrace{\left| \frac{e^{-i\frac{k^2}{2}h} - 1}{h} \right|^2}_{\xrightarrow{h\to 0} \frac{k^4}{4}} |\hat{\psi}_0(k)|^2 \, dk,$$

but dominated convergence only applies if $k^4 |\hat{\psi}_0(k)|^2$ is integrable, i.e., $k^2\hat{\psi}_0(k) \in L^2$.

$\Rightarrow$ $P_f(t)$ is strongly differentiable only as an operator on $H^2 := \{\psi\in L^2 : k^2\hat{\psi}_0(k) \in L^2\}$.

i.e., $P_f(t): H^2 \to L^2$

$\Leftrightarrow$ $\psi(t)$ is differentiable only for $\psi_0 \in H^2$.

And for $\psi_0 \in H^2$ we have

$$-\frac{1}{2}\underbrace{\Delta}_{\substack{\text{distributional}\\ \text{derivative}}} \psi(t) = -\frac{1}{2}\Delta \mathcal{F}^{-1} \underbrace{e^{-i\frac{k^2}{2}t}\mathcal{F}\psi_0}_{\hat{\psi}(t)} = \overbrace{\mathcal{F}^{-1}\frac{k^2}{2} \underbrace{e^{-i\frac{k^2}{2}t}\hat{\psi}_0}_{\hat{\psi}(t)}}^{\in L^2} = \overbrace{i\frac{d}{dt}\psi(t)}^{\in L^2}.$$

$\Rightarrow$ The free SE holds as equality of L^2 vectors.

Conclusion: For $\psi_0 \in H^2$, $\psi(t)$ solves the free Schrödinger equation $\forall t$ in the L^2 sense.
If $L^2 \ni \psi_0 \notin H^2$, then $\psi(t)$ solves the free Schrödinger equation in the sense of distributions only (as noted before).

We summarize the important properties of $P_f(t)$. These will provide a good framework to define propagators also for the interacting Schrödinger equation:

a) $P_f(t)$ is unitary $\forall t \in \mathbb{R}$.

b) P_f is strongly continuous.

c) P_f is a group homomorphism:

$$P_f(t) P_f(s) = \mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F} \mathcal{F}^{-1} e^{-i\frac{k^2}{2}s} \mathcal{F} = \mathcal{F}^{-1} e^{-i\frac{k^2}{2}(t+s)} \mathcal{F} = P_f(t+s) \ \forall s,t \in \mathbb{R}.$$

d) For $\psi_0 \in L^2$, $\psi(t) = P_f(t)\psi_0$ solves the free SE in the sense of distributions.

e) For $\psi_0 \in H^2 \subset L^2$, $\psi(t) = P_f(t)\psi_0$ solves the free SE in the L^2 sense.