Foundations of Mathematical Physics, Jacobs University, Fall 2021 — Session 20

Prof. Sören Petrat — Nov. 10, 2021

Last time: • For $A \in \mathcal{L}(V,W)$ we defined the adjoint $A' : W' \to V'$ by

$$(A'w')(v) = w'(Av) \quad \forall v \in V.$$

• Riesz Representation Thm.: $T \in \mathcal{H}' \Rightarrow T(\varphi) = \langle \psi_T, \varphi \rangle \ \forall \varphi \in \mathcal{H}$ for a unique $\psi_T \in \mathcal{H}$.

Today, we will discuss bounded generators, for which we have similar results as in finite dimensions. Next time we will sketch results for unbounded generators (as, e.g., relevant for the Schrödinger equation).

Riesz tells us that elements of $\mathcal{H}'$ can be be canonically identified with elements of $\mathcal{H}$:

Corollary 3.40:

$J : \mathcal{H} \to \mathcal{H}'$, $\psi \mapsto J\psi = \langle \psi, \cdot \rangle$ is a $\overbrace{\text{canonical}}^{\text{no arbitrary choices, e.g., of basis}}$ $\underbrace{\text{antilinear}}_{\text{due to antilinearity of the scalar product in the first variable}}$ $\overbrace{\text{bijection}}^{\text{by Riesz}}$ and a $\overbrace{\text{continuous}}^{\text{continuity of scalar product}}$ $\underbrace{\text{isometry}}_{\|J\psi\|_{\mathcal{L}(\mathcal{H},\mathbb{C})} = \|\psi\|_{\mathcal{H}}}$.

With that we can identify A' canonically with an operator A^* on $\mathcal{H}$:

Definition 3.41:

For $A \in \mathcal{L}(\mathcal{H})$, we define the Hilbert space adjoint $A^* : \mathcal{H} \to \mathcal{H}$, $A^* = \overbrace{J^{-1}}^{\mathcal{H}' \to \mathcal{H}} \overbrace{A'}^{\mathcal{H}' \to \mathcal{H}'} \overbrace{J}^{\mathcal{H} \to \mathcal{H}'}$.

Sometimes A^* is simply called "adjoint", or "Hermitian adjoint", and in the physics literature it is often denoted $A^\dagger$ ("A dagger").

Let us collect a few properties of A^*. First, with Riesz, we directly get

Proposition 3.42:

For $A \in \mathcal{L}(\mathcal{H})$ we have $\langle \psi, A\varphi \rangle = \langle A^*\psi, \varphi \rangle \; \forall \psi, \varphi \in \mathcal{H}$ and this property uniquely determines A^*.

Proof: By the definitions we have

$$\langle \psi, A\varphi \rangle = (J\psi)(A\varphi) = A'(J\psi)(\varphi) = JJ^{-1}A'J\psi(\varphi) = JA^*\psi(\varphi) = \langle A^*\psi, \varphi \rangle .$$

Also, $\varphi \mapsto \langle \psi, A\varphi \rangle$ is continuous and linear, so due to Riesz there is a unique $\eta \in \mathcal{H}$ s.t. $\langle \psi, A\varphi \rangle = \langle \eta, \varphi \rangle \; \forall \varphi \in \mathcal{H}$, so $\eta = A^*\psi$ is unique. $\square$

Before we continue, a few more standard properties and an example

Theorem 3.43: For $A, B \in \mathcal{L}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ we have

a) $(A+B)^* = A^* + B^*$, $(\lambda A)^* = \overline{\lambda} A^*$
b) $(AB)^* = B^* A^*$
c) $\|A^*\| = \|A\|$
d) $A^{**} = A$
e) $\|AA^*\| = \|A^*A\| = \|A\|^2$
f) $\ker A = (\operatorname{im} A^*)^\perp$ and $\ker A^* = (\operatorname{im} A)^\perp$

Proof: HW (a), b), c) follow directly from definition, d), e), f) are short computations)

As an example, consider the left and right shifts on ℓ^2:

The right shift is $T_r : \ell^2 \to \ell^2, (x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$. Then

$$\langle x, T_r y \rangle = \sum_{i=1}^{\infty} x_i (T_r y)_i = \sum_{i=2}^{\infty} x_i x_{i-1} = \sum_{i=1}^{\infty} x_{i+1} y_i =: \langle T_r^* x, y \rangle \text{, so } T_r^* = T_\ell \text{, where}$$

T_ℓ is the left shift $T_\ell : \ell^2 \to \ell^2, (x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$.

Note that T_r is isometric ($\|Tx\| = \|x\|$), but not surjective, so it is not unitary.
We have $T_r^* T_r = \mathrm{id}$, but $T_r T_r^* \neq \mathrm{id}$, so T_r^* is not the inverse of T_r (which isn't even invertible).

Based on this example, let us make the following nice connection to unitary operators:

$\boxed{\text{Proposition 3.45: } U \in \mathcal{L}(\mathcal{H}) \text{ is } \underbrace{\text{unitary}}_{\text{surjective + isometric}} \text{ if and only if } U^* = U^{-1}.}$

Proof: "$\Rightarrow$" We compute $\langle U^* U \psi - \psi, \varphi \rangle = \langle U^* U \psi, \varphi \rangle - \langle \psi, \varphi \rangle$

$$= \langle U\psi, U\varphi \rangle - \langle \psi, \varphi \rangle$$

$$= \langle \psi, \varphi \rangle - \langle \psi, \varphi \rangle$$

$$= 0 \quad \forall \psi, \varphi \in \mathcal{H} \text{, so } U^* U = \mathrm{id}$$

since U surjective $U U^* U = U$ implies $U U^* = \mathrm{id}$, so $U^{-1} = U^*$.

"$\Leftarrow$" If $U^* = U^{-1}$ then U is surjective.

Isometry? $\langle U\psi, U\varphi \rangle = \langle U^* U \psi, \varphi \rangle = \langle U^{-1} U \psi, \varphi \rangle = \langle \psi, \varphi \rangle$ ✓ □

Back to the adjoint. A nice class of bounded operators is the following:

$\boxed{\text{Definition 3.46: } A \in \mathcal{L}(\mathcal{H}) \text{ is called self-adjoint if } A^* = A.}$

So for $A \in \mathcal{L}(\mathcal{H})$ we have A self-adjoint $\overset{(3.42)}{\iff} \langle \psi, A\varphi \rangle = \langle A\psi, \varphi \rangle \; \forall \psi, \varphi \in \mathcal{H}$, i.e.,
A is symmetric

The difficulty for next time is that for unbounded operators symmetry does not imply self-adjointness.

Now we can make the connection to generators:

$$\boxed{\begin{array}{l}\underline{\text{Theorem 3.48:}}\ \text{Let } H \in \mathcal{L}(\mathcal{H}) \text{ be self-adjoint. Then } e^{-iHt} = \sum_{n=0}^{\infty} \frac{(-iHt)^n}{n!} \text{ defines} \\ \text{a unitary group with generator } H \text{ with domain } D(H) = \mathcal{H}. \text{ Moreover} \\ U: \mathbb{R} \to \mathcal{L}(\mathcal{H}),\ t \mapsto e^{-iHt} \text{ is (uniformly) differentiable.}\end{array}}$$

Proof: HW. Sketch:

- Well-definedness: show that $\sum_{n=0}^{k} \frac{(-iHt)^n}{n!}$ is a Cauchy sequence (in $\mathcal{L}(\mathcal{H})$ norm).
- Group property: direct computation.
- Unitarity: Carefully compute $\langle \varphi, e^{-iHt}\psi \rangle$ to find $(e^{-iHt})^*$; show it equals $(e^{-iHt})^{-1}$.
- Uniform differentiability: Need to check only at $t=0$ (why?). Estimate $\lim_{t \to 0} \left\| \frac{U(t) - \mathrm{id}_{\mathcal{H}}}{t} - (-iH) \right\|_{\mathcal{L}(\mathcal{H})}$.
- Schrödinger eq. clear from uniform differentiability.

So for bounded H, we can make sense of $e^{-iHt}\psi(0)$ being the solution to $i\frac{d}{dt}\psi(t) = H\psi(t)$.

For unbounded H (e.g., H containing differential operators) we will have a similar connection, but the definition of self-adjointness is more subtle.