Operations Research, Jacobs University, Fall 2021 — Session 2

Prof. Sören Petrat — Sep. 7, 2021

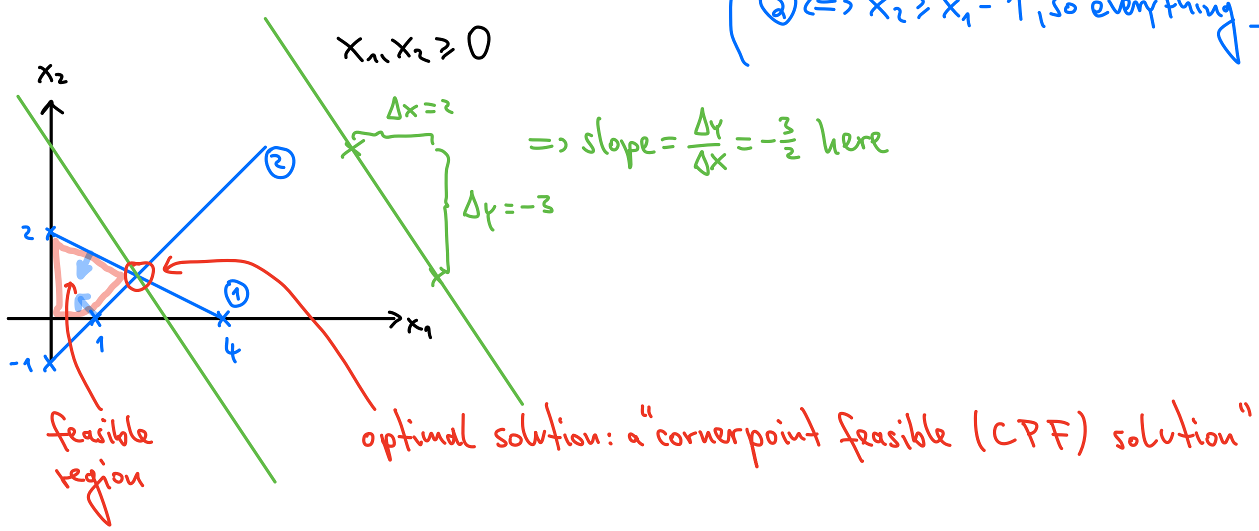
2. Linear Programming

2.1 Graphical Solutions

We consider examples to illustrate different possibilities for solutions.

1. Similar to introductory example:

- maximize $z = 3x_1 + 2x_2$ ($\Rightarrow$ line $x_2 = -\frac{3}{2}x_1 + \frac{z}{2}$, i.e., slope $-\frac{3}{2}$)
- with constraints
  $x_1 + 2x_2 \le 4$ ①   (note: ① $\Leftrightarrow x_2 \le 2 - \frac{x_1}{2}$, so everything under the line is allowed)
  $x_1 - x_2 \le 1$ ②   (② $\Leftrightarrow x_2 \ge x_1 - 1$, so everything above the line is allowed)
  $x_1, x_2 \ge 0$



At the $\boxed{\text{optimal corner point}}$ (where orange lines meet):
$$x_1 + 2x_2 = 4$$
$$x_1 - x_2 = 1$$

$\Rightarrow$ augmented matrix $\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & -1 & 1 \end{array}\right)$

Gaussian elimination: $R_1 - R_2 \to R_2$ $\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 3 & 3 \end{array}\right) \xrightarrow{R_2/3} \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow{R_1 - 2R_2 \to R_1} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right)$

$\Rightarrow$ solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, there $z = 3 \cdot 2 + 2 \cdot 1 = 8$.

$\Longleftrightarrow x_2 = -\frac{2}{3}x_1 + \frac{z}{9}$

2. • minimize $z = 6x_1 + 9x_2$ (slope $-\frac{2}{3}$)

• constraints:

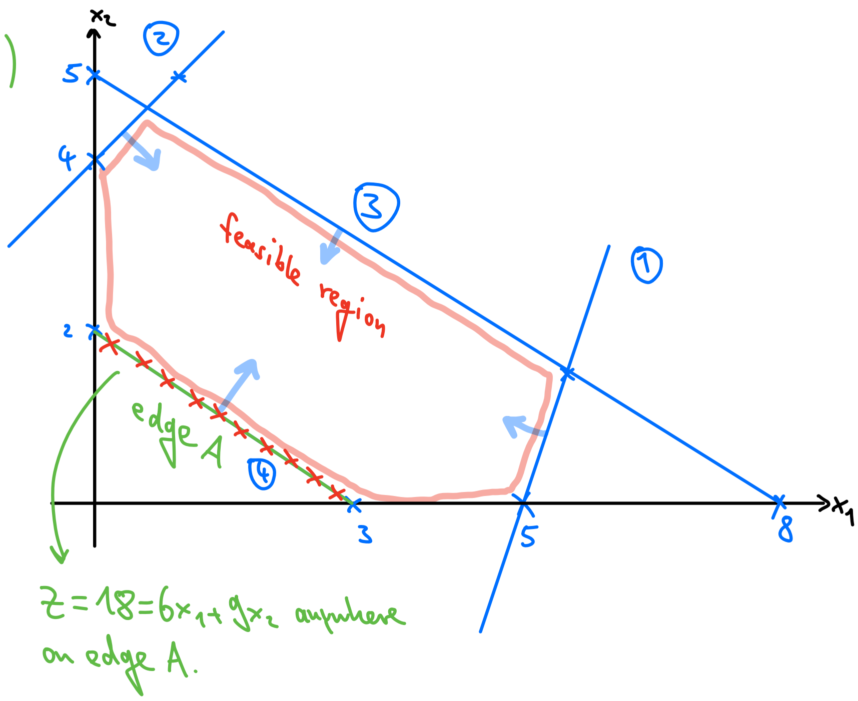
$3x_1 - x_2 \le 15$ (1)

$-x_1 + x_2 \le 4$ (2)

$5x_1 + 8x_2 \le 40$ (3)

$2x_1 + 3x_2 \ge 6$ (4)

$x_1, x_2 \ge 0$



$\Rightarrow$ Any point on edge A is an optimal solution, i.e., there are $\underbrace{\text{infinitely many}}$.

We call such problems $\boxed{\text{"degenerate"}}$.

meaning infinitely many points on the bounded line segment between points $(0,2)$ and $(3,0)$ (i.e., edge A)

(side note: recall that $x \ge y \Leftrightarrow 0 \ge y - x \Leftrightarrow -y \ge -x$
$\Leftrightarrow -x \le -y$)

3. • maximize $z = 4x_1 + 2x_2$

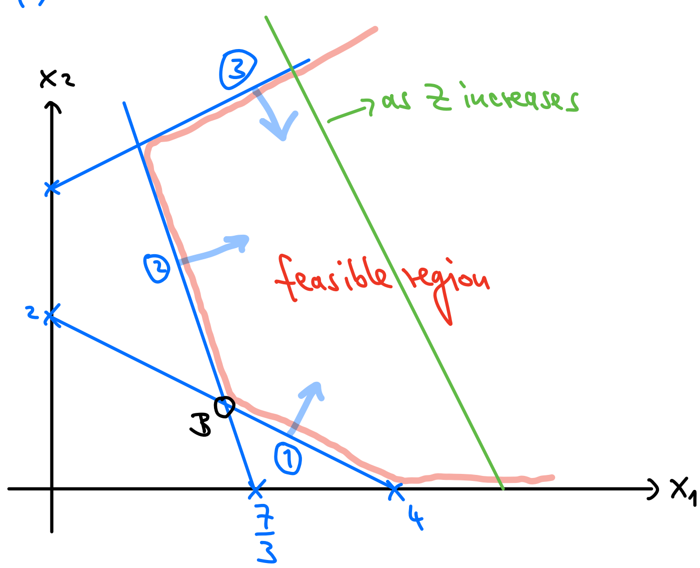
• constraints:

$x_1 + 2x_2 \ge 4$ (1)

$3x_1 + x_2 \ge 7$ (2)

$-x_1 + 2x_2 \le 7$ (3)

$x_1, x_2 \ge 0$



$\Rightarrow$ $\boxed{\text{Feasible region unbounded}}$, and z increases in unbounded direction.

$\Rightarrow$ There are feasible solutions, but we cannot find an optimal solution.

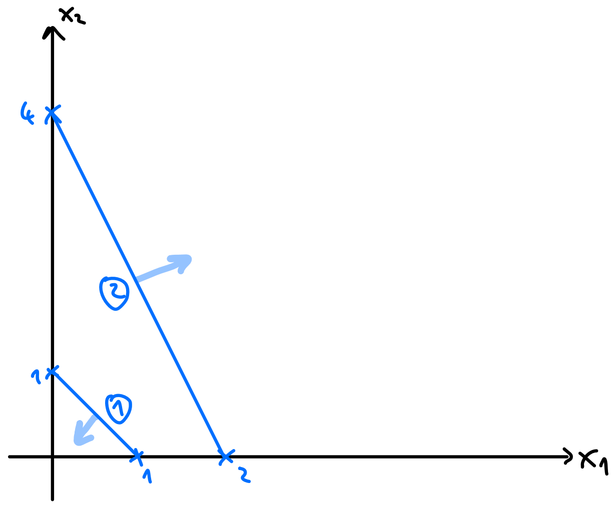
(Note: If z would be minimized, the optimal (CPF) solution would be at $B = \binom{2}{1}$, and $z = 10$.)

4. • maximize $z = 3x_1 + 4x_2$

• constraints: $x_1 + x_2 \leq 1$ ①

$2x_1 + x_2 \geq 4$ ②

$x_1, x_2 \geq 0$



$\Rightarrow$ The feasible region is empty; there are no feasible solutions.

We call such problems "over-constrained".

More generally, two prototypical examples (but mixtures are also possible) of Linear Programming (LP) models are:

I) Activity analysis problem:

- A = set of activities (or products)
- R = set of resources (or production facilities)
- w_{ij} = workload required from activity $i \in A$ on resource $j \in R$
- c_j = available capacity of resource $j \in R$
- p_i = profit from performing one unit of activity $i \in A$

- decision variables x_i: # of units of activity $i \in A$ to perform

LP problem: • maximize $z = \sum_{i \in A} p_i x_i$

• constraints: $\sum_{i \in A} w_{ij} x_i \leq c_j$ for all $j \in R$

$x_i \geq 0$ for all $i \in A$

II) Diet-type problem:

- F = set of foods
- N = set of nutrients
- c_i = unit cost of food $i \in F$
- r_j = minimum requirement for nutrient $j \in N$
- a_{ij} = amount of nutrient $j \in N$ from eating one unit of food $i \in F$
- decision variables x_i = # of units of food $i \in F$ to consume

$$\boxed{\begin{array}{lll} \text{LP problem:} & \bullet\ \text{minimize } z = \sum_{i \in F} c_i x_i & \\ & \bullet\ \text{constraint: } \sum_{i \in F} a_{ij} x_i \geq r_j & \text{for all } j \in N \\ & \qquad\qquad\qquad x_i \geq 0 & \text{for all } i \in F \end{array}}$$