2.2 Standard Form of LP Problems

Consider the following example of an LP problem:

- maximize $z = x_1 + 2x_2 + 3x_3$ $\overset{\text{Step 1}}{\Rightarrow}$ minimize $-x_1 - 2x_2 - 3x_3$
- constraints: $x_1 + x_2 - \overbrace{x_3}^{\text{Step 3: } x_3 = u - v} = 1$

$$\left.\begin{array}{l} -2x_1 + x_2 + 2x_3 \geq -5 \overset{\text{Step 1}}{\Rightarrow} 2x_1 - x_2 - 2x_3 \leq 5 \\ x_1 - x_2 \leq 4 \\ x_2 + x_3 \leq 5 \end{array}\right\} (*) \rightarrow \text{Step 2}$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

Claim: Every LP problem can be written in the $\boxed{\text{standard form}}$:

- minimize $c^T x$ $\left(= (c_1, \dots, c_m)\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}\right)$, with $c \in \mathbb{R}^m$
- subject to $Ax = b$ $\left(\Leftrightarrow \begin{pmatrix} A_{11} & \dots & A_{1m} \\ \vdots & & \\ A_{n1} & \dots & A_{nm} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}\right)$, with $A \in \overbrace{\mathrm{Mat}(n \times m)}^{\{n \times m \text{ matrices}\}}$, $b \in \mathbb{R}^n$

  and $x \geq 0$ (meaning $x_j \geq 0$ for all $j = 1, \dots, m$)

We illustrate this with the example above ("proof by example"):

Step 1: Turn maximization into minimization. Write inequalities in standard order ($\dots x_1 + \dots + \dots x_3 \leq \dots$). ← all variables on the left; $\leq$ sign; all numbers without variables on the right

Step 2: Turn inequalities into equalities + non-negativity constraints by introducing "slack variables":

(*) can be written as:
$$2x_1 - x_2 - 2x_3 + s_1 = 5 \quad \text{with } s_1 \geq 0$$
$$x_1 - x_2 + s_2 = 4 \quad \text{with } s_2 \geq 0$$
$$x_2 + x_3 + s_3 = 5 \quad \text{with } s_3 \geq 0$$

Step 3: Replace variables without non-negativity constraint by differences:

$$x_3 = u - v \quad \text{with } u \geq 0, v \geq 0$$

To summarize, we have rewritten the problem in standard form with:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ u \\ v \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}, \quad A = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & u & v & s_1 & s_2 & s_3 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 2 & -1 & -2 & 2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad b = \begin{pmatrix} 1 \\ 5 \\ 4 \\ 5 \end{pmatrix}, \quad c = \begin{pmatrix} -1 \\ -2 \\ -3 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We are now confronted with solving a system of linear eq.s $Ax = b$, with $A \in \mathrm{Mat}(n \times m)$, $b \in \mathbb{R}^n$.

Note:

- As in the example above, for us A is typically a wide matrix $(m > n)$, i.e., the system is underdetermined and there are many solutions.
- In Finite Mathematics you learned about least-norm solutions, i.e., solutions that minimize $\|x\|$. Our goal is: Find solution that optimizes the linear objective function.

How do we find solutions?

↳ Use Gaussian elimination to bring augmented matrix into row echelon form.

Ex.: $A = \begin{pmatrix} 1 & 3 & 1 & 1 \\ 2 & 6 & 0 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\Rightarrow$ augmented matrix: $\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 2 \\ 2 & 6 & 0 & -1 & 1 \end{array}\right)$

row-echelon form: $-2R_1 + R_2 \rightarrow R_2$: $\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 2 \\ 0 & 0 & -2 & -3 & -3 \end{array}\right)$

$R_2/-2$: $\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 2 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \end{array}\right)$

$R_1 - R_2 \rightarrow R_1$: $\left(\begin{array}{cccc|c} 1 & 3 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \end{array}\right)$

pivots

$$\Rightarrow \begin{aligned} x_1 + 3x_2 \quad - \tfrac{1}{2} x_4 &= \tfrac{1}{2} \\ x_3 + \tfrac{3}{2} x_4 &= \tfrac{3}{2} \end{aligned}$$

, i.e., we have two "free" variables, e.g., $x_4 = -\mu$, $x_2 = -\lambda$

$\Rightarrow x_3 = \frac{3}{2} + \frac{3}{2}\mu$, $x_1 = \frac{1}{2} + 3\lambda - \frac{1}{2}\mu$

$$\Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + 3\lambda - \frac{1}{2}\mu \\ -\lambda \\ \frac{3}{2} + \frac{3}{2}\mu \\ -\mu \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{3}{2} \\ 0 \end{pmatrix}}_{\text{particular solution}} + \underbrace{\lambda \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{3}{2} \\ -1 \end{pmatrix}}_{\text{the two vectors span the space of solutions to the homogeneous equation } Ax = 0}$$

Recipe to get solutions directly from augmented matrix in row echelon form:

- add zero rows (s.t. 1's are on diagonal)
- put -1 on diagonal in the zero rows
- read off solution

here:
$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & -1 & 0 \end{array}\right)$$

$$\Rightarrow \text{solution} = \lambda \cdot \begin{pmatrix} \vdots \end{pmatrix} + \mu \cdot \begin{pmatrix} \vdots \end{pmatrix} + \begin{pmatrix} \vdots \end{pmatrix}$$

Note: Here $B = \{1, 3\}$ are the linearly independent columns (i.e., the columns with the pivots).

For the particular solution (also: basic solution): $x_j = 0$ for $j \notin B$

But: there are many ways to parametrize the solutions, e.g., also:

lin. indep. columns

$$\left(\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & -\frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & -1 & 0 \end{array}\right) \quad (R_1 \text{ above}/3) \qquad \text{i.e., } B = \{2, 3\}$$

$$\Rightarrow x = \begin{pmatrix} 0 \\ \frac{1}{6} \\ \frac{3}{2} \\ 0 \end{pmatrix} + \tilde{\lambda} \begin{pmatrix} -1 \\ 1/3 \\ 0 \\ 0 \end{pmatrix} + \tilde{\mu} \begin{pmatrix} 0 \\ -\frac{1}{6} \\ \frac{3}{2} \\ -1 \end{pmatrix}$$

another possibility:

$B = \{2, 4\}$

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & -\frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 1 & 1 \end{array}\right) \xrightarrow{\frac{1}{6}R_4 + R_2 \to R_2} \begin{array}{c} \begin{array}{ccccc} \text{add } -1 & & \text{add } -1 & & \\ \downarrow & & \downarrow & & \end{array} \\ \left(\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & \frac{1}{9} & 0 & \frac{1}{3} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 1 & 1 \end{array}\right) \end{array}$$

$$\Rightarrow x = \begin{pmatrix} 0 \\ \frac{1}{6} \\ 0 \\ 1 \end{pmatrix} + \tilde{\lambda} \begin{pmatrix} -1 \\ \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} + \tilde{\tilde{\mu}} \begin{pmatrix} 0 \\ \frac{1}{9} \\ -1 \\ \frac{2}{3} \end{pmatrix}$$