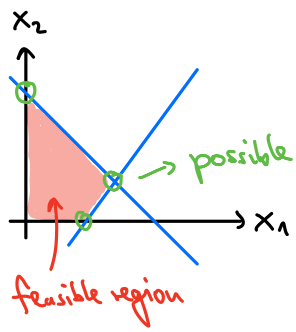
Operations Research, Jacobs University, Fall 2021 — Session 4

Prof. Sören Petrat — Sep. 14, 2021

Last time: standard form of LP problems:
- minimize $z = c^T x$
- constraints: $Ax = b$, $x \geq 0$

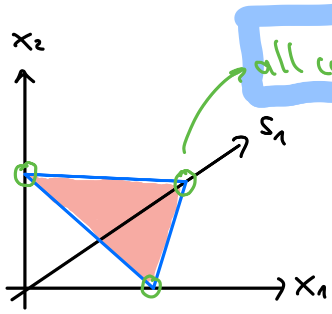
How does the feasible region look like?

- in our first examples (not standard form): polygon



Intuition: if there is an optimal solution, there is at least one which is CPF.

- in standard form:



$\boxed{\text{all corner points are basic feasible solutions}}$

basic → at least one component is 0

Intuition: if there is an optimal solution, at least one should be basic.

Last time: with Gaussian elimination we can find basic solutions:

these are the particular solutions to $Ax = b$ with $x_j = 0$ for $j \notin B$

B = set of columns with pivots

Let us now prove our intuition from above:

Theorem: If a standard form LP problem has optimal solutions, then there is an optimal basic solution (i.e., an optimal solution that is a vertex/corner of the feasible region).

Proof: We suppose x is an optimal solution which is not basic.

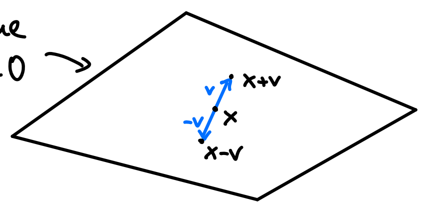
i.e., we are not already at the boundary

Let us assume that all components of x are non-zero (otherwise we disregard/delete the 0 components, which does not change the problem).

Idea: Now we shift x without changing the value of the objective function until we reach a basic solution ("hit the boundary").

Since $Ax=b$ is a linear constraint and $x>0$, there must be at least one direction vector $v \neq 0$, so that $x+v$ and $x-v$ are still feasible.

at least one component $=0$ on this boundary



We have:

- $A(x+v)=b \Rightarrow Ax+Av=b$, and since $Ax=b$, we get $Av=0$
- since x is optimal: $c^Tx \leq c^T(x+v)$, i.e., $c^Tv \geq 0$ ($c^T(x+v) = c^Tx + c^Tv$)

  $c^Tx \leq c^T(x-v)$, i.e., $-c^Tv \geq 0 \Rightarrow c^Tv \leq 0$

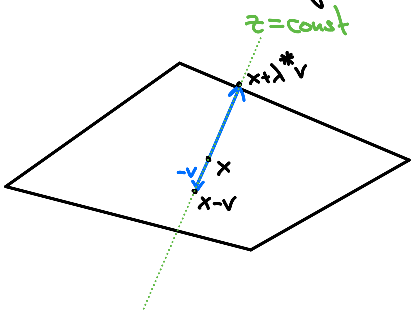
$\Rightarrow c^Tv = 0$ ($x \pm v$ does not change value of objective function!)

Now let v have at least one negative component (otherwise take $-v$ instead of v).

↳ this is to make sure we actually go towards a corner

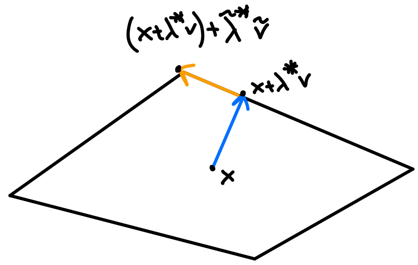
Then $x+\lambda v$ for $\lambda \in [0,1]$ is feasible and $c^T(x+\lambda v) = c^T x + \lambda \underbrace{c^T v}_{=0} = c^T x$, so $x+\lambda v$ is also optimal.

Now increase λ: at some value $\lambda = \lambda^*$, one component of $x+\lambda v$ will change sign from + to −, thus leaving the feasible region.



$\Rightarrow x+\lambda^* v$ is still feasible and optimal, and has at least one component $=0$.

Finally, we repeat this until solution is basic.



$\square$

Conclusion: We need to check only basic feasible solutions.