Operations Research, Jacobs University, Fall 2021 — Session 9

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Last time: We solved an LP problem first, and then defined shadow prices.

Today: How to compute shadow prices directly (via solving the "dual" LP problem).

Recall the example from last time: maximize profit $z = 3x_1 + 2x_2 = c^T x$

with constraints
$$\left.\begin{aligned} 5x_1 &\le 100 \\ 10x_2 &\le 100 \\ 4x_1 + 3x_2 &\le 100 \\ 3x_1 + 5x_2 &\le 100 \end{aligned}\right\} Ax \le b$$
$$x_1, x_2 \ge 0$$

Now: Consider the following scenario: A company wants to buy our production capacity.

What are fair prices y_1, y_2, y_3, y_4 for the resources (1), (2), (3), (4)?

In our example:
- profit per car: 3
- profit per truck: 2
- current car assembly hours: 5 for constraint (1), 4 for constraint (3), 3 for constraint (4)
- trucks: 10 for (2), 3 for (3), 5 for (4)

Thus we want:
$$\left.\begin{aligned} &5y_1 + 4y_3 + 3y_4 \ge 3 \\ &\underbrace{10y_2 + 3y_3 + 5y_4}_{=A^T y} \ge \underbrace{2}_{c} \end{aligned}\right\}$$
selling capacity to produce one car/truck needs to be at least as profitable as producing a car/truck

(Recall: $A = \begin{pmatrix} 5 & 0 \\ 0 & 10 \\ 4 & 3 \\ 3 & 5 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 5 & 0 & 4 & 3 \\ 0 & 10 & 3 & 5 \end{pmatrix}$ is the transpose of A.)

The price for all capacity is $y_1 \cdot 100 + \ldots + y_4 \cdot 100 = b^T y$. Minimizing this yields the minimum prize.

This leads to the $\boxed{\text{"dual problem"}}$: • minimize $b^T y$,
• subject to $A^T y \geq c$ and $y \geq 0$.

as compared to the original $\boxed{\text{"primal problem"}}$: • maximize $c^T x$,
• subject to $Ax \leq b$ and $x \geq 0$.

Two results about the relation between dual and primal LP:

- Note that $c^T x = x^T c \underset{c \leq A^T y}{\leq} x^T A^T y \underset{(AB)^T = B^T A^T}{=} (Ax)^T y \underset{Ax \leq b}{\leq} b^T y$.

  This is known as $\boxed{\text{weak duality}}$:
  $\boxed{\text{If } x \text{ is a solution to the primal problem (i.e., } x \text{ is feasible, but not necessarily optimal), and } y \text{ is a solution to the dual problem, then } c^T x \leq b^T y.}$

- A bit harder to prove (but intuitively clear) is $\boxed{\text{strong duality}}$:
  $\boxed{\text{The dual has an optimal solution if and only if the primal does. In this case } c^T x = b^T y.}$