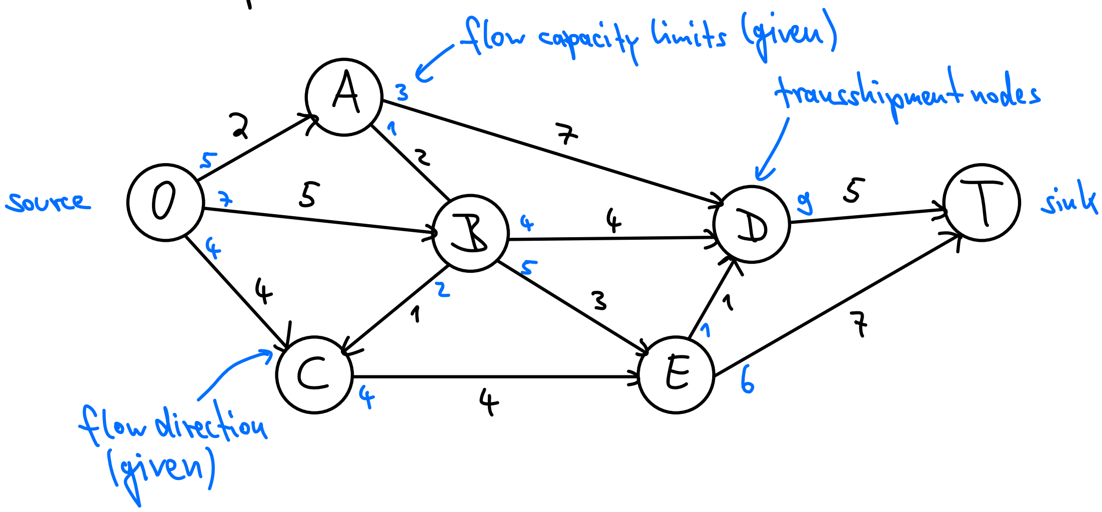
Operations Research, Jacobs University, Fall 2021 — Session 13

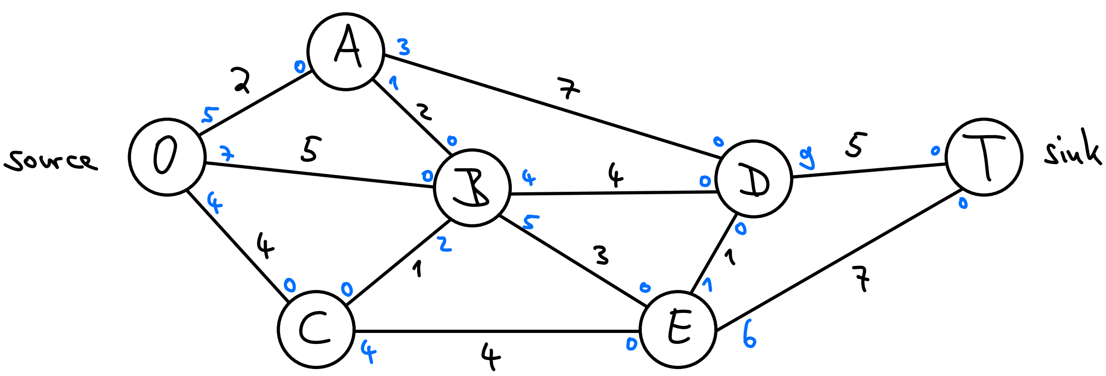
Prof. Sören Petrat — Oct. 14, 2021

- Maximum Flow problem:



Objective: maximize flow from source to sink

Augmented Path algorithm: Draw network as



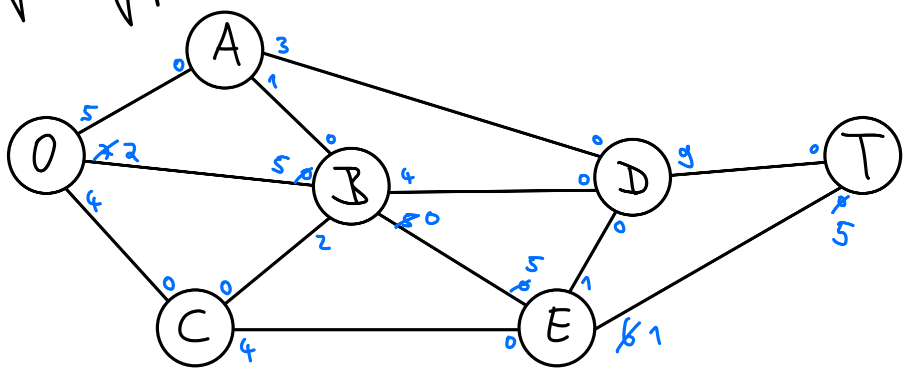
Augmenting path: directed path from source to sink s.t. every arc has strictly positive capacity

Now:
- choose an augmenting path
- increase flow by residual capacity
- repeat until no augmenting path can be chosen anymore

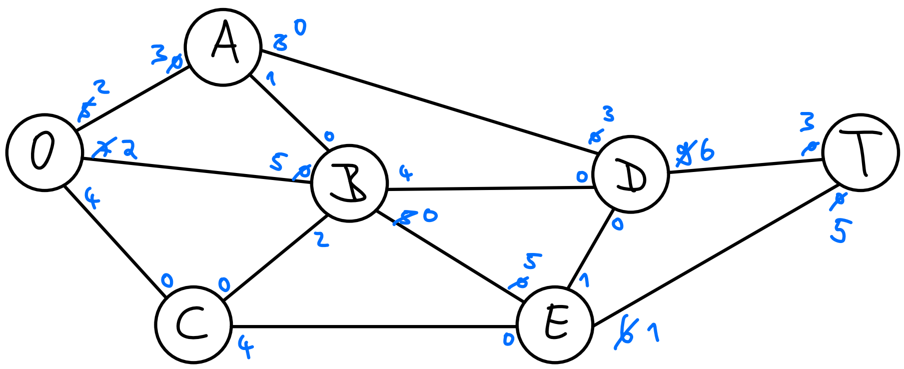
in picture: smallest possible number at beginning of arcs

For our example:

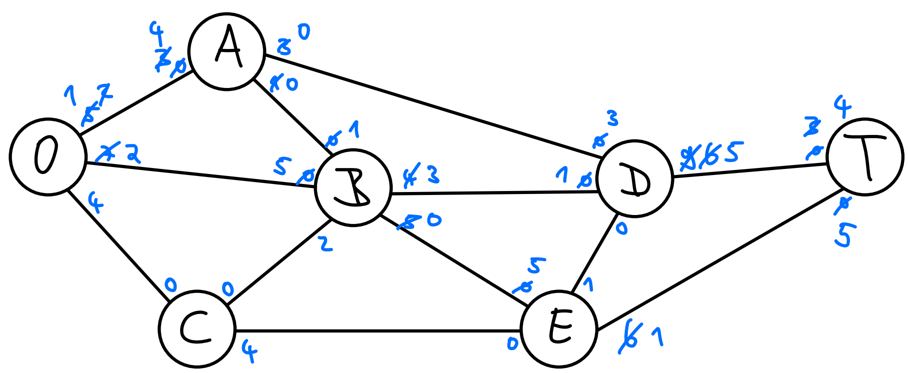
A possible augmenting path is 0-B-E-T: residual capacity: 5 (B-E)



arbitrary next choice: 0-A-D-T: res. cap.: 3 (A-D)



0-A-B-D-T: res. cap.: 1 (A-B)

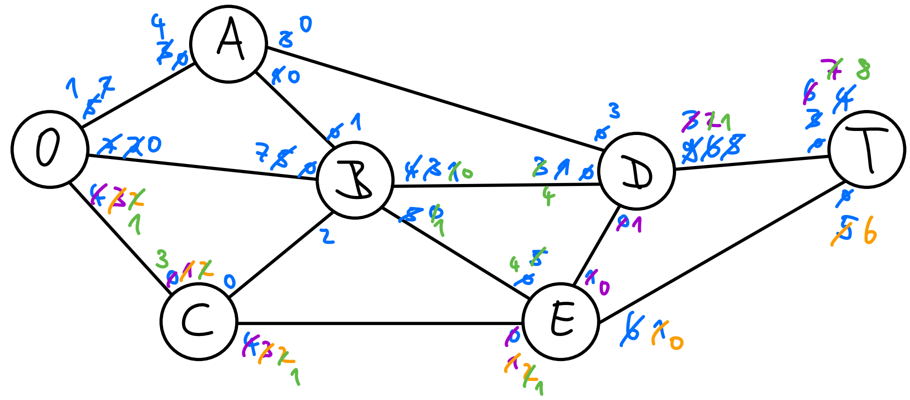


0-B-D-T: res. cap.: 2 (0-B)

0-C-E-D-T: res. cap.: 1 (E-D)

0-C-E-T: res. cap.: 1 (E-T)

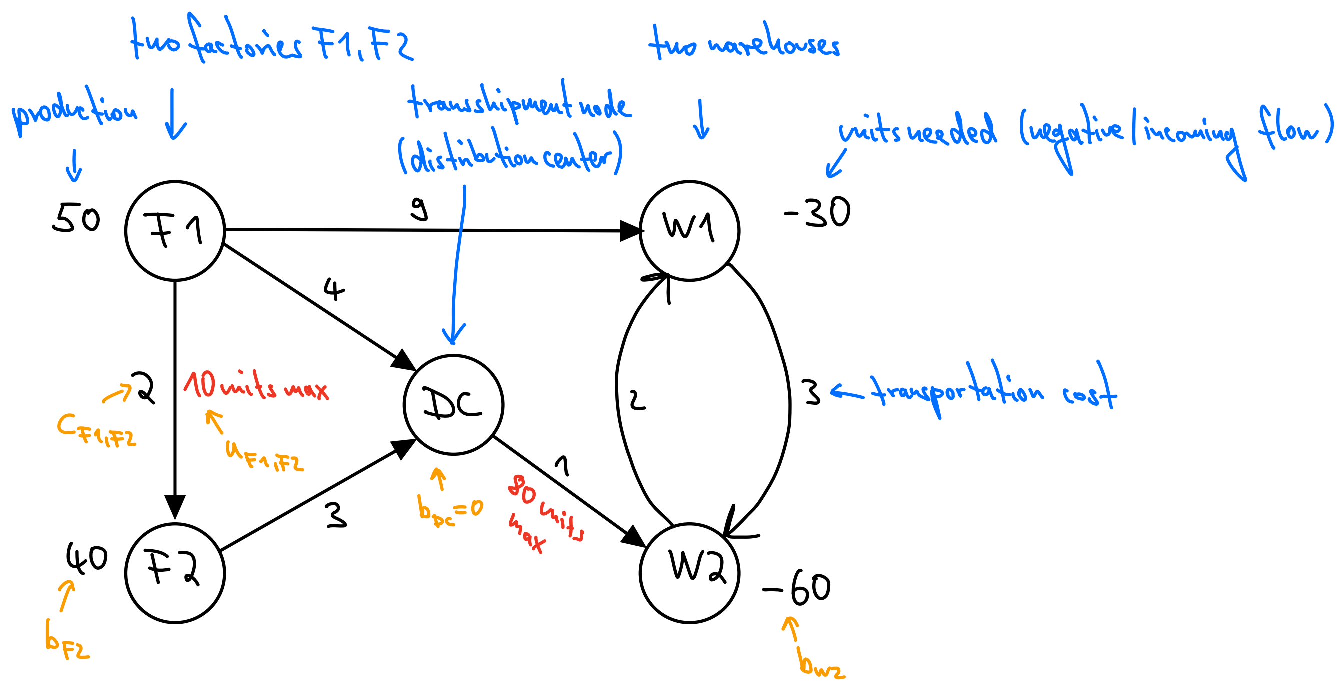
0-C-E-B-D-T: res. cap.: 1 (B-D)



=> No more augmenting paths, we have found an optimal solution: 8+6 = 14 trips can be made from O to T (more details can be read off from final picture).

More generally, all the previous 3 problem types can be formulated as **minimum cost flow problems.**

Example (Hillier, Liebermann Chapters 3.4 and 9.6):



General formulation:
- nodes $i \in N$
- directed arcs $(i,j) \in A$
- c_{ij}: unit cost of transportation on arc (i,j)
- u_{ij}: max. capacity on arc (i,j)
- node constraints
  - $b_i > 0$ for supply/source nodes
  - $b_i < 0$ for demand/sink nodes
  - $b_i = 0$ for transshipment nodes
- x_{ij}: flow from i to j (decision variables)

LP formulation:
- Minimize cost $Z = \sum_{i,j} c_{ij} x_{ij}$
- Constraints: $\underbrace{\sum_j x_{ij}}_{\text{outgoing flow at node } i} - \underbrace{\sum_j x_{ji}}_{\text{incoming flow at node } i} = b_i$ for all nodes $i \in N$

  and $0 \leq x_{ij} \leq u_{ij}$ for all arcs $(i,j) \in A$

Note: Similarly as discussed before:
- One can show that a necessary condition for feasible solutions is $\sum_i b_i = 0$ (supply = demand). This can always be achieved by introducing dummy nodes (similarly as we discussed before).
- All basic variables in all basic feasible solutions are integer, if all b_i and u_{ij} are integer.
- A faster network simplex method is available.

How can our previous cases be formulated as min. cost flow problems?

- Transportation problem: – only supply and demand nodes (no transshipment nodes)
  – all $u_{ij} = \infty$ since no upper bound constraints

- Shortest Path problem: – origin = supply node with $b_O = 1$
  – destination = demand node with $b_T = -1$
  – other nodes are transshipment
  – draw all arcs in both directions (except source/sink)
  – all $u_{ij} = \infty$

- Max Flow problem: – all $c_{ij} = 0$
  – source $b_O = \overline{F}$ large, sink $b_T = -\overline{F}$, all other nodes $b_i = 0$
  – u_{ij} as given
  – extra arc from source to sink with $c_{OT} = M$ very large (and $u_{OT} = \infty$)