

We continue the example from last time

# of medical teams	Countries		
	1	2	3
0	0	0	0
1	45	20	50
2	70	45	70
3	90	75	80
4	105	110	100
5	120	150	130

← # of additional person-years of life saved (in thousands)

(note: not proportional to # of teams!)

Goal: distribute 5 medical teams to the three countries s.t. maximum person-years saved.

Note: This problem type is called "distribution of effort problem".

Dynamic Programming formulation:

• stages = countries, x_i = # of teams sent to country i , s = # of teams still available

• here: $f(s, x_i) = b_{s, x_i} + f_{i+1}^*(s - x_i)$
 ↪ see table above # of teams available in next step (the new s in the next step)

Solution:

$i=3$:
 ↗
 country 3

s	f_3^*	x_3^*
0	0	0
1	50	1
2	70	2
3	80	3
4	100	4
5	130	5

$i=2:$

s	$f_2(s, x_2) = b_{s_1 x_2} + f_2^*(s - x_2)$						$f_2^*(s)$	x_2^*
	$x_2=0$	$x_2=1$	$x_2=2$	$x_2=3$	$x_2=4$	$x_2=5$		
0	0+0	—	—	—	—	—	0	0
1	0+50	20+0	—	—	—	—	50	0
2	0+70	20+50	45+0	—	—	—	70	0 or 1
3	0+80	20+70	45+50	75+0	—	—	95	2
4	0+100	20+80	45+70	75+50	110+0	—	125	3
5	0+130	20+100	45+80	75+70	110+50	150+0	160	4

$i=1:$

s	$f_1(s, x_1) = b_{s_1 x_1} + f_2^*(s - x_1)$ <small>$\Rightarrow s$ here</small>						$f_1^*(s)$	x_1^*
	$x_1=0$	$x_1=1$	$x_1=2$	$x_1=3$	$x_1=4$	$x_1=5$		
5	0+160 =160	45+125 =170	70+95 =165	90+70 =160	105+50 =155	120+0 =120	170	1

\Rightarrow Solution: send 1 team to country 1 (4 remaining), 3 to country 2 (1 remaining), and 1 to country 3, to save the maximal 170 thousand person-years of life.

Example: Local job shop problem (Hillier, Lieberman: Chapter 11.3)

Setup: different seasons with minimal worker requirements:

Season	Spring	Summer	Autumn	Winter	Spring	...
Requirement	255	220	240	200	255	

Too much employment costs 2000\$ per person per season.

Changing employment from one season to the next costs 200\$. $(\text{difference in employment})^2$.

We assume fractional employment is possible (part-time work). (i.e., optimal solution need not be integer.)

Goal: Find hiring schedule that minimizes costs.

- We introduce:
- Stage 4 = spring, Stage 3 = winter, Stage 2 = autumn, Stage 1 = summer
 - x_n = employment level for stage n . Note: $x_4 = 255$.
 - r_n = minimum requirements from table above.

\Rightarrow Feasible values for x_n : $r_n \leq x_n \leq 255$

Note: The states $s_n = x_{n-1}$ can now take a continuum of values.

Similar to before, we set

$$f_n(s_n, x_n) = \underbrace{200(x_n - s_n)^2}_{\text{employment change cost}} + \underbrace{2000(x_n - r_n)}_{\text{extra employment cost}} + \underbrace{f_{n+1}^*(x_n)}_{\text{optimal costs for later stages}} \quad \left. \vphantom{f_n(s_n, x_n)} \right\} \text{cost given employment } s_n \text{ at stage } n-1, \text{ and } x_n \text{ at stage } n, \text{ and optimal future cost}$$

$$f_n^*(s_n) = \min_{r_n \leq x_n \leq 255} f_n(s_n, x_n) \quad \left. \vphantom{f_n^*(s_n)} \right\} \text{optimal cost given employment } s_n \text{ at stage } n-1, \text{ and optimal in the future}$$

Solution:

$$\text{Stage } n=4: \quad \begin{array}{c|c|c} s_4 & f_4^*(s_4) & x_4^* \\ \hline 200 \leq s_4 \leq 255 & 200(255 - s_4)^2 & 255 \end{array}$$

$$\begin{aligned} \text{Stage } n=3: \quad f_3^*(s_3) &= \min_{200 \leq x_3 \leq 255} f_3(s_3, x_3) \\ &= \min_{200 \leq x_3 \leq 255} \left[200(x_3 - s_3)^2 + 2000(x_3 - 200) + \underbrace{200(255 - x_3)^2}_{f_4^*(x_3)} \right] \end{aligned}$$

We find the minimum by setting the partial derivative w.r.t. x_3 to zero (keeping s_3 fixed):

$$\frac{\partial f_3(s_3, x_3)}{\partial x_3} = 400(x_3 - s_3) + 2000 - 400(255 - x_3) = 400(2x_3 - s_3 - 250) \stackrel{!}{=} 0$$

$$\Rightarrow x_3^* = \frac{250 + s_3}{2}$$

Since $\frac{\partial^2 f_3(s_3, x_3)}{\partial x_3^2} = 800 > 0$, we indeed have a minimum.

Possible s_3 : $240 \leq s_3 \leq 255 \Rightarrow 245 \leq x_3^* \leq 252.5$ is feasible ($200 \leq x_3 \leq 255$), so x_3^* is always the feasible minimum.

$$\begin{aligned} \Rightarrow f_3^*(s_3) &= 200(x_3^* - s_3)^2 + 2000(x_3^* - 200) + 200(255 - x_3^*)^2 \\ &= 200 \left(\frac{250 + s_3}{2} - s_3 \right)^2 + 2000 \left(\frac{250 + s_3}{2} - 200 \right) + 200 \left(255 - \left(\frac{250 + s_3}{2} \right) \right)^2 \\ &= \underbrace{200 \left(\frac{250 - s_3}{2} \right)^2}_{= \frac{1}{4}(250 - s_3)^2} + \underbrace{2000 \left(\frac{250 + s_3}{2} - 200 \right)}_{= \frac{-150 + s_3}{2}} + \underbrace{200 \left(255 - \left(\frac{250 + s_3}{2} \right) \right)^2}_{= \left(\frac{260 - s_3}{2} \right)^2} \\ &= \frac{1}{4}(250 - s_3)^2 \end{aligned}$$

$$\Rightarrow \begin{array}{c|c|c} s_3 & f_3^*(s_3) & x_3^* \\ \hline 240 \leq s_3 \leq 255 & 50(250 - s_3)^2 + 50(260 - s_3)^2 + 1000(s_3 - 150) & \frac{250 + s_3}{2} \end{array}$$

One proceeds similarly with stages 2 and 1, but we skip the details.

The results are:

$$n=2: \begin{array}{c|c|c} s_2 & f_2^*(s_2) & x_2^* \\ \hline 220 \leq s_2 \leq 240 & 200(240 - s_2)^2 + 115,000 & 240 \\ 240 \leq s_2 \leq 255 & \frac{200}{9} [(240 - s_2)^2 + (255 - s_2)^2 + (270 - s_2)^2] + 2000(s_2 - 195) & \frac{2s_2 + 240}{3} \end{array}$$

$$n=1: \begin{array}{c|c|c} s_1 & f_1^*(s_1) & x_1^* \\ \hline 255 & 185,000 & 247.5 \end{array}$$

$\Rightarrow x_1^* = 247.5, x_2^* = 245, x_3^* = 247.5, x_4^* = 255$, and minimum cost is 185,000 \$.