Operations Research, Jacobs University, Fall 2021 Session 20

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3.3 Inventory Theory

Inventory management is very important in the business world e.g., for

- retail
- factories (materials for production need to be available)

General ideas:

- Costs for storing ("carrying") inventory, but also for resupplying
- First, we look at deterministic models, where the demand is known (e.g., production). After, we look at stochastic models, where demand is a random variable.

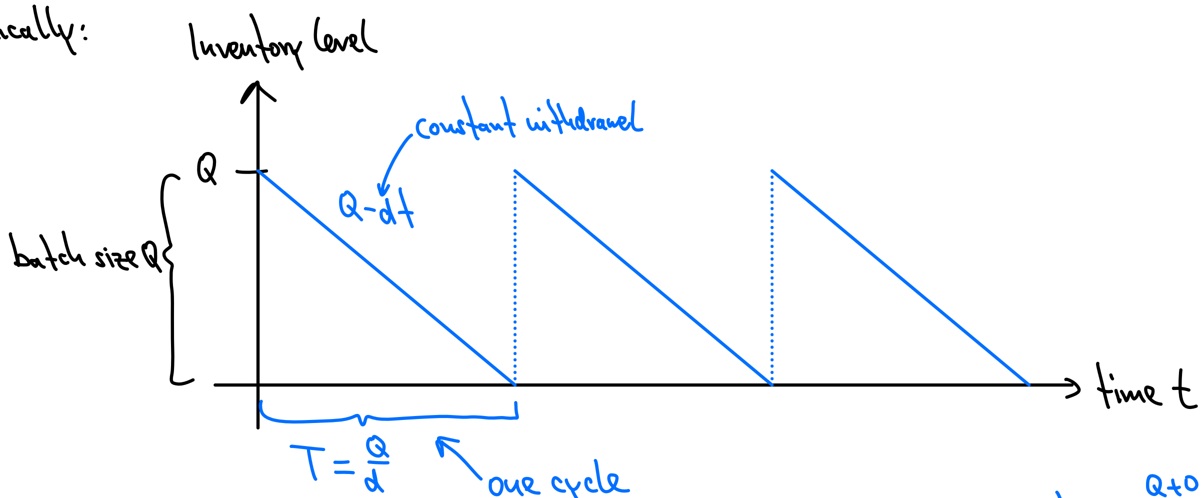
First, let us consider inventory management under the following assumptions:

- The cost of ordering is:
  - K setup costs per order,
  - c unit costs.
- The holding (or storage) cost is h per unit per time in inventory.
- There is a constant withdrawal rate of d units per time.
- We do not allow for shortages.
- There is continuous review, i.e., inventory level is continuously checked (as opposed to periodic checks)

These assumptions lead to the basic economic order quantity model (EOQ).

Note: Under these assumptions, it is always optimal that new orders arrive exactly when inventory is empty.

Graphically:



The cost per cycle is then $C_{cycle} = \underbrace{K + cQ}_{\text{order costs}} + \underbrace{h \overset{\text{average inventory} = \frac{Q+0}{2} = \frac{Q}{2}}{\frac{Q}{2}} T}_{\substack{\text{holding costs} \\ = h \cdot (\text{average inventory}) \cdot (\text{time})}}$

$$\Rightarrow \text{Total cost per time } C = \frac{C_{cycle}}{T} = \frac{K + cQ + h\frac{Q}{2}T}{T} = \frac{K+cQ}{T} + h\frac{Q}{2} = \frac{K+cQ}{Q/d} + h\frac{Q}{2}$$

$$\Rightarrow C = \frac{dK}{Q} + dc + h\frac{Q}{2}$$

What is the optimal order quantity Q^* that minizes cost per time C?

→ We need to find the minimum:

$$\frac{dC}{dQ} = -\frac{dK}{Q^2} + \frac{h}{2} \overset{!}{=} 0 \quad \Rightarrow \quad Q^* = \sqrt{\frac{2dK}{h}} \qquad \text{(EOQ formula)}$$

The corresponding optimal cycle time is $T^* = \frac{Q^*}{d} = \sqrt{\frac{2K}{dh}}$

This basic EOQ model applies to the following Speakers example (Hillier, Liebermann: Chapter 18.1):

- 12 000 \$ setup cost for producing a batch of speakers
- 10 \$ cost for producing one speaker
- 0.30 \$ holding costs per speaker per month (storage space, but also costs of tied up capital)
- speakers are used for continuous production of TVs, $d = 8000$ per month

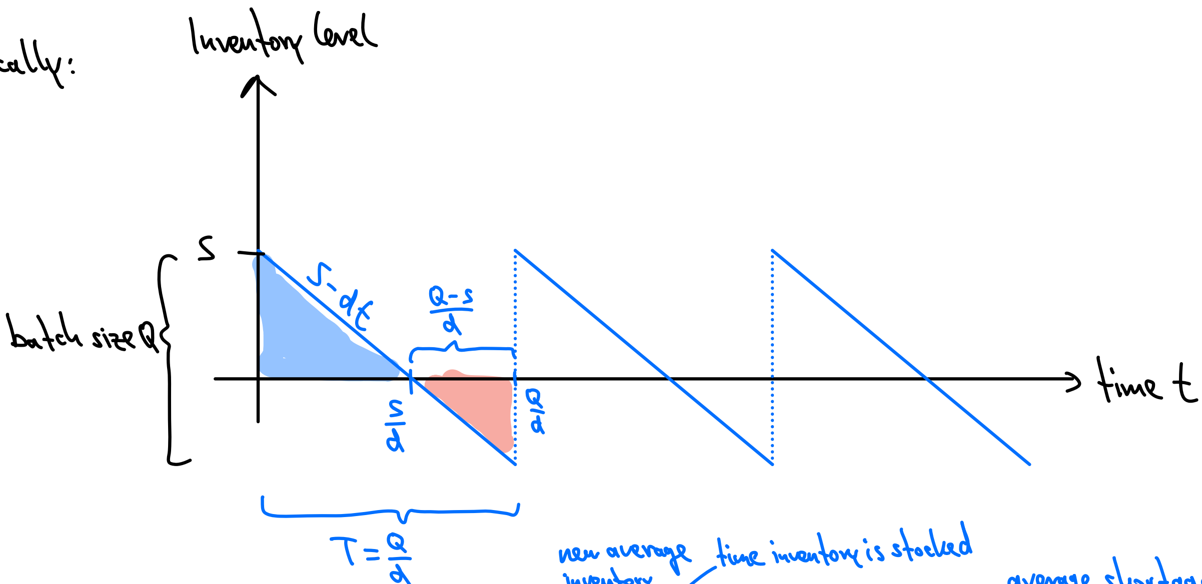
Then $Q^* = \sqrt{\frac{2 \cdot 8000 \cdot 12000}{0.3}} = 25298$ units should be produced every

$T^* = \frac{25298}{8000} \approx 3.2$ months.

Next: Let us assume the inventory can be empty for part of a cycle at a penalty p per unit per time. Withdrawels that cannot be fulfilled will be postponed and processed when new batch arrives.

This leads to the EOQ model with planned shortages

Graphically:



new average inventory

time inventory is stocked

average shortage

$$\Rightarrow \text{Cycle cost } C_{cycle} = \underbrace{K + cQ}_{\text{order cost}} + \underbrace{\boxed{h \frac{S}{2} \frac{S}{d}}}_{\text{holding cost}} + \underbrace{\boxed{p \frac{Q-S}{2} \frac{Q-S}{d}}}_{\text{penalty}}$$

shortage time

$$\Rightarrow \text{Total cost per time } C = \frac{C_{cycle}}{T} = \frac{K}{Q/d} + \frac{cQ}{Q/d} + \frac{1}{2}\frac{hS^2}{d\,Q/d} + \frac{1}{2}p\frac{(Q-S)^2}{d\,Q/d}$$

$$= \frac{dK}{Q} + dc + \frac{1}{2}\frac{hS^2}{Q} + \frac{1}{2}p\underbrace{\frac{(Q-S)^2}{Q}}_{=\frac{(Q-S)}{Q}(Q-S) = (1-\frac{S}{Q})(Q-S)}$$

Here, Q and S are decision variables, so we need to compute two partial derivatives:

$$\frac{\partial C}{\partial S} = \frac{hS}{Q} - p\frac{(Q-S)}{Q} \overset{!}{=} 0 \quad \Rightarrow \underbrace{hS = p(Q-S)}_{(*)} \quad \Rightarrow (h+p)S = pQ$$

$$\Rightarrow S = \frac{p}{h+p}Q$$

$$\frac{\partial C}{\partial Q} = -\frac{dK}{Q^2} - \frac{1}{2}\frac{hS^2}{Q^2} + \frac{1}{2}p\left(\frac{S}{Q^2}(Q-S) + \underbrace{1-\frac{S}{Q}}_{=\frac{Q-S}{Q}}\right)$$

$$= -\frac{dK}{Q^2} - \frac{1}{2}\frac{hS^2}{Q^2} + \frac{1}{2}\underbrace{p(Q-S)}_{=hS \text{ (see Equation } (*))}\left(\frac{S}{Q^2} + \frac{1}{Q}\right) \overset{!}{=} 0 \qquad (*)$$

$$\Rightarrow (*) \Rightarrow \quad -\frac{dK}{Q^2} - \frac{1}{2}\frac{hS^2}{Q^2} + \frac{1}{2}hS\left(\frac{S}{Q^2} + \frac{1}{Q}\right) = 0$$

$$\Leftrightarrow \quad -\frac{dK}{Q^2} + \frac{1}{2}\frac{hS}{Q} = 0$$

$S = \frac{p}{h+p}Q$

$$\Rightarrow \quad \frac{dK}{Q^2} = \frac{1}{2}\frac{hp}{h+p}$$

$$\Rightarrow \quad Q^* = \sqrt{\frac{2dK}{h}}\sqrt{\frac{h+p}{p}} \quad \text{is the minimum}$$

with corresponding $S^* = \frac{p}{h+p}Q^* = \sqrt{\frac{2dK}{h}}\sqrt{\frac{p}{h+p}}$,

and cycle time $T^* = \frac{Q^*}{d} = \sqrt{\frac{2K}{dh}}\sqrt{\frac{p}{h+p}}$.

Note: If $p \to \infty$, then $\sqrt{\frac{h+p}{p}} \to 1$, and we recover the basic EOQ model from before.