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Next, let us consider stochastic models for perishable products, also called "newsvendor problem".

We consider/assume:

- A single perishable product, e.g., newspaper, food, flowers, seasonal goods such as clothing (but also, e.g., airline reservations)
- single time period
- at the end of period, product has salvage value (e.g., selling clothes out of season at a discount)
- no initial inventory
- decision variable y = # of items to stock
- the demand D is a random variable (we will need to make reasonable assumptions for its probability distribution)

- K = setup cost, irrelevant here (exactly one order is placed)
- c = unit cost of purchasing/producing
- h = holding cost per item = cost of storage − salvage value
- p = shortage cost (penalty) per item, e.g., lost revenue or lost customer goodwill

The amount sold is $\min\{D, y\} = \begin{cases} D & \text{if } D < y, \\ y & \text{if } D \geq y. \end{cases}$

The cost is $C(D,y) = \underbrace{cy}_{\text{order cost}} + \underbrace{p\max\{0, D-y\}}_{\text{penalty}} + \underbrace{h\max\{0, y-D\}}_{\text{holding cost}}$

penalty $= \begin{cases} 0 & \text{if } D < y, \\ p(D-y) & \text{if } D > y. \end{cases}$

holding cost $= \begin{cases} 0 & \text{if } D > y \text{ (all sold)} \\ h(y-D) & \text{if } D < y \text{ (leftovers)} \end{cases}$

Goal: minimize expected cost, given some probability distribution $\underbrace{P_D(d)}_{\text{probability that demand} = d}$ for the demand.

Expected cost $\mathbb{E}[C](y) = \sum_{d=0}^{\infty} C(d,y)\, P_D(d)$.

How do we model $P_D(d)$?

Possibility (1): brute-force using empirical data, i.e., P_D = empirical probability distribution

E.g., suppose in the last 20 days, we sold the following # of products:

9 15 14 10 7 9 8 3 12 18 5 20 16 17 7 10 12 8 9 12

Then:

# sold:	3	5	7	8	9	10	12	14	15	16	17	18	20
frequency: (how many days?)	1	1	2	2	3	2	3	1	1	1	1	1	1
empirical $P_D = \frac{\text{frequency}}{\text{total \#}}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$

Then $\mathbb{E}[C] = C(3)\frac{1}{20} + C(5)\frac{1}{20} + C(7)\frac{2}{20} + \dots$

Problems:
- might not have enough data (e.g., 1, 2, 4, 6 items sold never occured above)
- historical data not always good

Possibility (2): use a theoretic $P_D(d)$ (maybe using additionally mean or spread of historical data)

If d ranges over large number of values, it makes sense to approximate it by a continuous probability distribution $\varphi(d)$.

Then
$$\mathbb{E}[C](y) = \int_0^\infty C(x,y)\,\varphi(x)\,dx$$
$$= \int_0^\infty \left[cy + p\max\{0, x-y\} + h\max\{0, y-x\} \right] \varphi(x)\,dx$$
$$= cy \underbrace{\int_0^\infty \varphi(x)dx}_{\substack{=1,\text{ since}\\ \varphi \text{ is a probability}\\ \text{distribution}}} + p \underbrace{\int_0^\infty \max\{0, x-y\}\varphi(x)dx}_{= \int_y^\infty (x-y)\varphi(x)dx} + h \underbrace{\int_0^\infty \max\{0, y-x\}\varphi(x)dx}_{= \int_0^y (y-x)\varphi(x)dx}.$$
$$= cy + p\int_y^\infty (x-y)\varphi(x)dx + h\int_0^y (y-x)\varphi(x)\,dx$$

Goal: minimize $\mathbb{E}[C](y)$. Thus we compute:
$$\frac{d\,\mathbb{E}[C](y)}{dy} = c + \underbrace{p\int_y^\infty (-\varphi(x))\,dx - p(x-y)\varphi(x)\Big|_{x=y}}, + h\int_0^y \varphi(x)dx + h(y-x)\varphi(x)\Big|_{x=y}$$

Note: $\frac{d}{dy}\int_y^\infty f(x,y)dx = \frac{d}{dy}\left[F(\infty,y) - \underbrace{F(y,y)}_{= F(x_1,x_2)|_{x_1=x_2=y}}\right]$ (F anti-derivative in first variable, i.e., $\frac{\partial}{\partial x}F(x,y) = f(x,y)$)
$$= \frac{\partial}{\partial y}F(\infty,y) - \underbrace{\frac{\partial}{\partial x_1}F(x_1,y)\Big|_{x_1=y}}_{=f(y,y)} - \frac{\partial}{\partial x_2}F(y,x_2)\Big|_{x_2=y}$$
$$= \int_y^\infty \frac{\partial}{\partial y}f(x,y)dx - f(y,y)$$

$$\Rightarrow \frac{d\,\mathbb{E}[C](y)}{dy} = c - p\int_y^\infty \varphi(x)dx + h\int_0^y \varphi(x)dx$$

Let us introduce the cumulative distribution function $\Phi(y) := \int_0^y \varphi(x)\,dx$.
Note that $\Phi(\infty) = \int_0^\infty \varphi(x)\,dx = 1$ (all probabilities integrate up to 1).

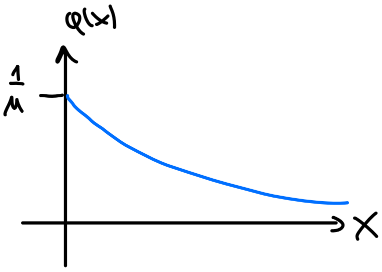
$\Phi(y)$ tells us the probability that the demand is satified if we order y items.

Then
$$\frac{d\,\mathbb{E}[C](y)}{dy} = c - p\Big(\underbrace{\int_0^\infty \varphi(x)\,dx}_{=1} - \underbrace{\int_0^y \varphi(x)\,dx}_{=\Phi(y)}\Big) + h\underbrace{\int_0^y \varphi(x)\,dx}_{=\Phi(y)}$$
$$= c - p + (p+h)\,\Phi(y) \overset{!}{=} 0$$

$$\Rightarrow \boxed{\Phi(y^*) = \frac{p-c}{p+h}}$$
i.e., we should choose y^* s.t. this equation is satified. → solutions can be found algebraically or numerically / graphically

$\Phi(y^*)$ is called "optimal service level".

Example: Assume exponential distribution $\varphi(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}$, with $\mu > 0$ the mean value.



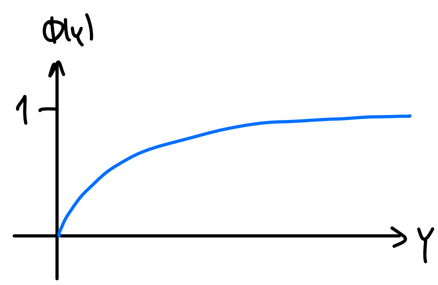
Note: μ is indeed the mean, since

change of variables: $\frac{x}{\mu} = y$

$$\mathbb{E}(X) = \int_0^\infty x\,\varphi(x)\,dx = \int_0^\infty \frac{x}{\mu} e^{-\frac{x}{\mu}}\,dx = \mu \int_0^\infty y\,e^{-y}\,dy$$

integration by parts
$$= \mu\underbrace{\left[-y e^{-y}\right]_0^\infty}_{=0} + \mu\int_0^\infty e^{-y}\,dy = -\mu e^{-y}\Big|_0^\infty = 0 - (-\mu) = \mu$$

Then $\Phi(y) = \int_0^y \frac{1}{\mu} e^{-\frac{x}{\mu}}\,dx = -e^{-\frac{x}{\mu}}\Big|_0^y = -e^{-\frac{y}{\mu}} + 1$



$$\Phi(y) = \frac{p-c}{p+h} \iff 1 - e^{-\frac{y}{\mu}} = \frac{p-c}{p+h} \iff e^{-\frac{y}{\mu}} = 1 - \frac{(p-c)}{p+h} = \frac{p+h}{p+h} - \frac{(p-c)}{p+h} = \frac{c+h}{p+h}$$

natural logarithm

$\ln\frac{a}{b} = -\ln\frac{b}{a}$

$$\Rightarrow -\frac{y}{\mu} = \ln\frac{c+h}{p+h} \Rightarrow y = -\mu \ln\frac{c+h}{p+h} = \mu \ln\frac{p+h}{c+h}$$

$\Rightarrow$ For exponential distribution with mean μ, the optimal order quantity is $\boxed{y^* = \mu \ln\frac{p+h}{c+h}}$.

Numerical example: For $\mu = 10\,000$, $c = 200$, $p = 450$, $\underbrace{h = -90}_{\text{a large salvage value can make } h \text{ negative.}}$, we find

$y^* \approx 11\,856$ (so 1856 items more than the average should be stocked).

Note that $\underbrace{\Phi(y^*) = \frac{450-200}{450-90} = 0.694}_{\text{optimal service level}}$ i.e., the demand is satisfied with 69.4% probability here.