Operations Research, Jacobs University, Fall 2021 — Session 23

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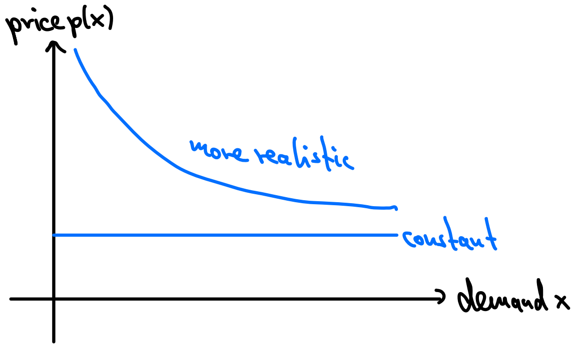
## 3.4 Nonlinear Programming

A general nonlinear programming problem has the standard form:

Maximize $f(x)$ subject to $g_i(x) \leq b_i$ for $i = 1, \dots, m$, and $x \geq 0$, where $x = (x_1, \dots, x_n)$.

A few examples how non-linear $f$ or $g_i$ can arise:

- Wyndor Glass Co. (product mix problem): unit profit often not fixed, but there is a price-demand curve, e.g.,




Then profit $Z(x) = x\,p(x) - \underbrace{c}_{}x$

unit production cost: sometimes also a function of $x$

- Portfolio selection for investments: Suppose $n$ stocks (or other securities) are considered, $x_j$ = # of shares of stock $j$ in portfolio. Suppose stock $j$ has average return $\mu_j$, then the expected value of portfolio is $R(x) = \sum_{j=1}^{n} \mu_j x_j$. But we should also try to minimize risk. If $\sigma_{ij}$ are the covariances ($\sigma_{jj}$ = variance of stock $j$), then we should also minimize $V(x) = \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j$. Then, e.g., we could set up the nonlinear programming problem:
  Minimize $V(x)$ subject to $\sum_{j=1}^{n} \mu_j x_j \geq L$ ($L$ = minimal acceptable expected return) and $\sum_{j=1}^{n} P_j x_j \leq B$, with $P_j$ = price of stock $j$, $B$ = budget, $x \geq 0$.

Next, let us look at specific examples that highlight the differences to linear Programming.

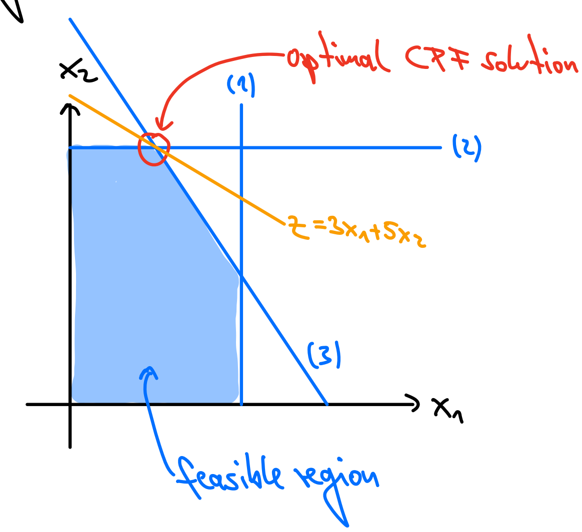
Let us use Wyndor Glass Co. as underlying example. Recall the LP version:

- Maximize $Z = 3x_1 + 5x_2$,
- subject to $x_1 \leq 4$ (1)

  $2x_2 \leq 12$ (2)

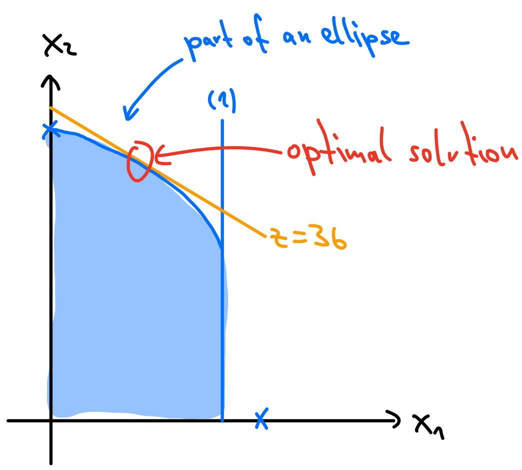
  $3x_1 + 2x_2 \leq 18$ (3)

  $x_1, x_2 \geq 0$




Nonlinear variations:

(1) Nonlinear constraint: replace (2) and (3) by $9x_1^2 + 5x_2^2 \leq 216$




Optimal solution is $(2, 6)$:

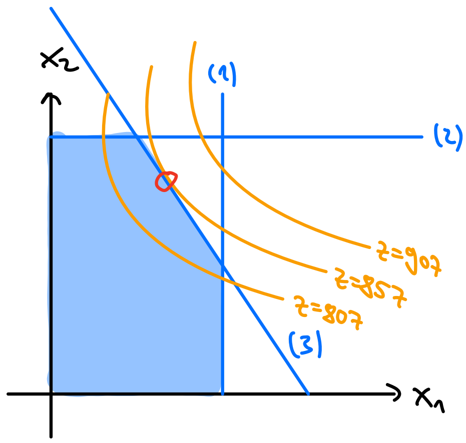
↳ at the boundary of feasible region

↳ but not a cornerpoint anymore

⇒ simplex method (which goes through CPF solutions only) does not work anymore

(2) Nonlinear objective function, linear constraints:

Maximize $Z = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2$ subject to (1), (2), (3).




Optimal solution is at boundary of feasible region, but not at a cornerpoint.

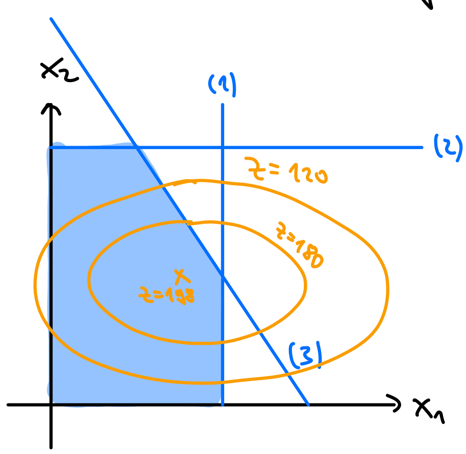
(3) Maximize $Z = 54x_1 - 9x_1^2 + 78x_2 - 13x_2^2$ subject to (1), (2), (3).

Let us check what the maximum of $Z$ without constraints is:

$$\frac{\partial Z}{\partial x_1} = 54 - 18x_1 \overset{!}{=} 0 \Rightarrow x_1 = 3$$

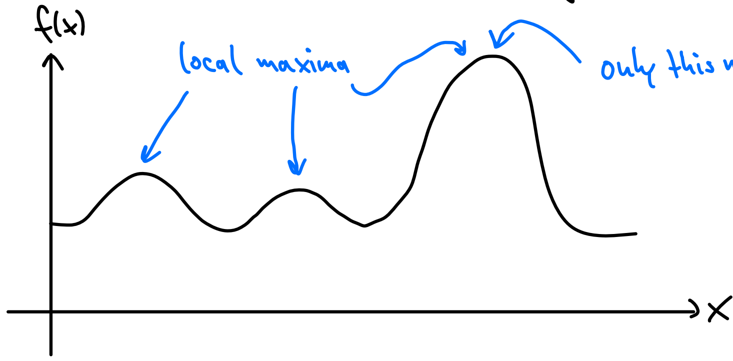
$$\frac{\partial Z}{\partial x_2} = 78 - 26x_2 \overset{!}{=} 0 \Rightarrow x_2 = 3$$

Since $Z(x_1, x_2)$ is a downward open paraboloa, there is only one maximum $(x_1, x_2) = (3, 3)$. This is within the feasible region. Here, $Z = 198$.




=> Optimal solution might be inside feasible region!

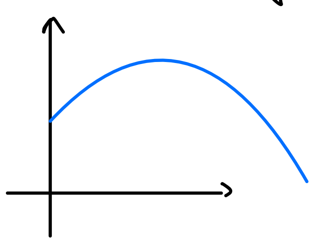
Another problem: A local maximum might not be a global maximum



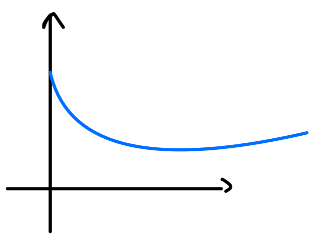

Nonlinear Programming algorithms are generally unable to distinguish between local and global maxima, so they might get stuck in a local maximum (and not find the global one).

But there are reasonable sufficient conditions for a maximum to be global:

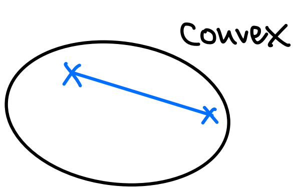
(i) For maxima: objective function is always "curving downward", i.e., concave:

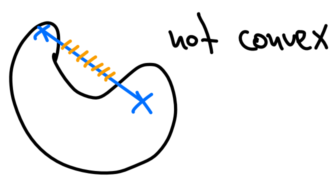


For minima: objective function is always "curving upward", i.e., convex:



(ii) The feasible region is convex, i.e., for any two points in the feasible region, the line segment between them is also in the feasible region.



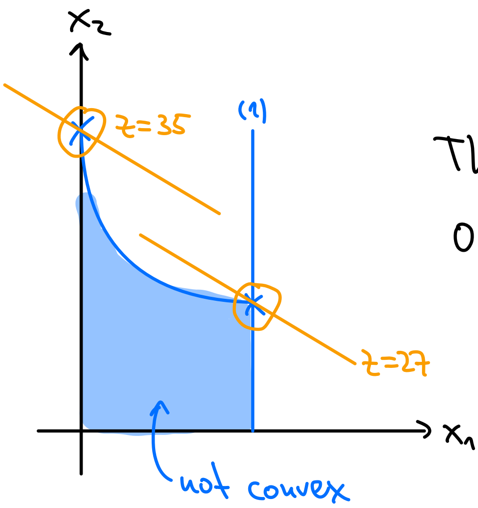

For such convex optimization problems, solvers such as ipopt work well.

(4) Feasible region not convex: • Maximize $Z = 3x_1 + 5x_2$,

• subject to $x_1 \le 4$,

$$8x_1 - x_1^2 + 14x_2 - x_2^2 \le 49,$$

$$x_1, x_2 \ge 0.$$




There are two local maxima: (4,3) and (0,7).

Only (0,7) is global, but algorithm might get stuck at (4,3).