Operations Research, Jacobs University, Fall 2021 — Session 24

Prof. Sören Petrat — Nov. 25, 2021

Today, let us discuss different types of nonlinear programming problems.

(1) Unconstrained optimization.

Objective: maximize $f(x)$ over all $x \in \mathbb{R}^n$. (No constraints.)

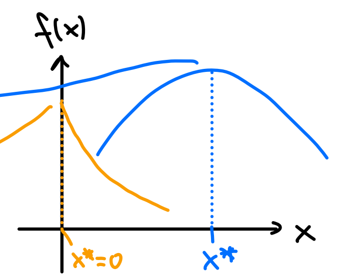
Assuming f is differentiable, then a necessary condition for $x = x^*$ to be optimal is

$$\frac{\partial f}{\partial x_j} = 0 \quad \text{at } x = x^* \text{ for all } j = 1, \dots, n.$$

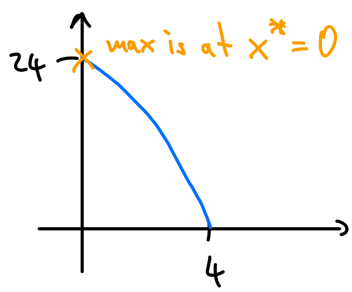
If f is additionally concave, then this condition is also sufficient.

If there are nonnegativity constraints $x \geq 0$ (i.e., $x_1, \dots, x_n \geq 0$), then the condition becomes

$$\frac{\partial f}{\partial x_j} \begin{cases} = 0 & \text{if } x_j^* > 0 \\ \leq 0 & \text{if } x_j^* = 0 \end{cases}$$



E.g., $f(x) = 24 - 2x - x^2$:



max is at $x^* = 0$, where derivative is not zero:

$$\frac{df}{dx} = -2 - 2x = -2 \leq 0 \text{ at } x = 0$$

Note: $\frac{df}{dx} = 0$ can be solved numerically if no analytic solution can be found.

Good numerical algorithms are the bisection method and Newton's method (or combinations of both).

(2) General nonlinear programs with differentiable constraint and objective functions.

Here, one can derive necessary condition for x^* to be an optimal solution. These are called KKT (Karush-Kuhn-Tucker) conditions.
(See Chapter 12.6 in Hillier, Lieberman.)

(3) Quadratic programming.

quadratic objective fct.

Maximize $f(x) = c^T x - \frac{1}{2} x^T Q x$, $c \in \mathbb{R}^n$, $Q \in \mathbb{R}^n \times \mathbb{R}^n$ ($n \times n$ matrix)
subject to $Ax \leq b$, $x \geq 0$.

linear constraints

Here, the KKT conditions lead to a linear programming problem with an additional "complementarity constraint" (nonlinear), for which a modified simplex method is available.

(4) Convex programming.

Maximize $f(x)$ for concave f,
subject to $g_i(x) \leq b_i$, $x \geq 0$ for convex g_i.

Then the KKT conditions are also sufficient. Then also solvers such as ipopt work well.

(5) Fractional programming.

Maximize $f(x) = \frac{f_1(x)}{f_2(x)}$. Here, some special methods are available. Let us discuss linear fractional programming: Maximize $f(x) = \frac{c^T x + c_0}{d^T x + d_0}$ with $c, d \in \mathbb{R}^n$, $c, d \geq 0$, $c_0, d_0 \in \mathbb{R}$, $d_0 \geq 0$,

subject to $Ax \leq b$, $x \geq 0$.

We can write $f(x) = c^T \underbrace{\frac{x}{d^Tx+d_0}}_{=:y} + c_0 \underbrace{\frac{1}{d^Tx+d_0}}_{=:t} = c^Ty + c_0 t$,

and $Ax \leq b \iff A\frac{x}{d^Tx+d_0} \leq b\frac{1}{d^Tx+d_0} \iff Ay \leq bt$.

The new decision variables are related by

$$d^Ty + d_0 t = d^T\frac{x}{d^Tx+d_0} + d_0\frac{1}{d^Tx+d_0} = \frac{d^Tx+d_0}{d^Tx+d_0} = 1.$$

Thus, we only need to solve the LP problem:

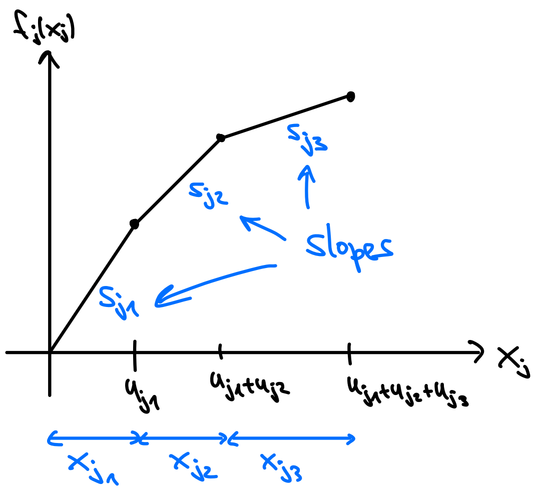
Maximize $f(y,t) = c^Ty + c_0 t$,

subject to $Ay \leq bt$, $d^Ty + d_0 t = 1$, and $y, t \geq 0$.

(6) Separable programming.

Here $f(x) = \sum_{j=1}^{n} f_j(x_j)$, i.e., the contributions to objective fct. (e.g., profit) from each acitivity x_j are independent.

A special case is when each f_j is piecewise linear:



Idea: Write $x_j = x_{j1} + x_{j2} + x_{j3}$

with $0 \leq x_{j1} \leq u_{j1}$, $0 \leq x_{j2} \leq u_{j2}$, $0 \leq x_{j3} \leq u_{j3}$

Special restriction: $x_{j(i+1)} = 0$ if $x_{ji} < u_{ji}$, e.g.

$\underbrace{x_{j2} = 0 \text{ if } x_{j1} < u_{j1}}_{\text{if } x_{j1} < u_{j1} \text{ only } x_{j1} \text{ is non-zero}}$, $x_{j3} = 0$ if $x_{j2} < u_{j2}$.

Then $f_j(x_j) = s_{j1}x_{j1} + s_{j2}x_{j2} + s_{j3}x_{j3}$.

This is an LP problem, up to the special restriction (that can be dealt with).

Note: piecewise linear functions can be used to approximate any (preferably concave) function: