Operations Research, Jacobs University, Fall 2021 — Session 25

Prof. Sören Petrat — Nov. 30, 2021

Final exam: close to HW/midterm problems, with slight variations and slightly different numbers

We go through HW problems/practice exams on Thursday.

Today: class summary

1. Introduction

↳ What is OR?
- Scientific/mathematical approach to management decisions.
- Optimization applied to industrial, logistical, organizational problems.
- In class we have developed lots of tools for various problems from different domains.

↳ General OR workflow: Def. problem, gather data, formulate math problem, solve it (then sensitivity analysis and recommendation/implementation)

2. Linear Programming

Central important method of OR ($\frac{1}{2}$ to $\frac{2}{3}$ of this class).

Lots of problems can be recast as or approximated by LP problems!

2.1 Graphical Solutions

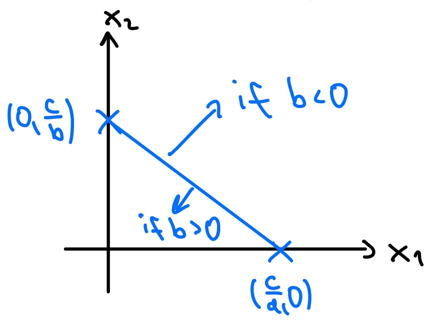
↳ Only works for simple LP problems (2 or 3 decision variables)

↳ For us it was more a tool to understand possible shapes of the feasible region (for LP but also nonlinear problems) and the simplex method

Tips for drawing the feasible region:

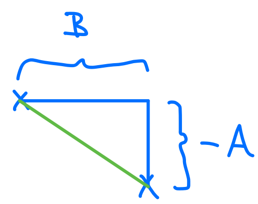
- Constraint $ax_1 + bx_2 \leq c$

  $\Rightarrow$ draw line $ax_1 + bx_2 = c$: it goes through $(0, \frac{c}{b})$ and $(\frac{c}{a}, 0)$



- Objective fct. $Z = Ax_1 + Bx_2$

  $\Rightarrow$ draw line with slope $-\frac{A}{B}$:



Key points:
- Practice finding optimal solutions graphically
- Be aware of different possible shapes of feasible region
- Central role of CPF solutions

2.2 Standard Form of LP Problems

↳ Convention: Minimize $Z = c^T x$ $(c \in \mathbb{R}^m)$,

subject to $Ax \leq b$ $(A \in Mat(n \times m), b \in \mathbb{R}^n)$, and $x \geq 0$.

(n constraints, m decision variables, usually $m > n$)

Main result: If this has optimal solutions, a $\underbrace{\text{basic solution}}$ is among them.

This is the foundation of the simplex method! → at least one component $= 0$

2.3 The Simplex Method

↳ Maybe most fundamental algorithm in OR:
- Start with basic feasible sol.
- Choose leaving and entering variables as discussed
- Repeat until no further improvement possible or some problem occurs (e.g., unbounded feasible region)

↳ Practice using a simplex tableau to solve LP problems

↳ But: in practice (large-scale problems), the simplex algorithm is used on computers.
There is many free and commercial software available.
We used pyomo because:
- it is integrated in python, which is widely used, free, and has many packages for specialized tasks
- fast, easy to use

⟹ Study how to read pyomo programs and their output!

2.4 The Dual LP Problem

Starting from the standard form above, it is: Minimize $b^T y$,
subject to $A^T y \geq c$ and $y \geq 0$.

↳ Math results:
- $c^T x \leq b^T y$, where x is sol. to primal problem, y to dual problem (weak duality)
- Primal has optimal sol. ⟺ Dual has optimal sol.
  In this case $c^T x = b^T y$ (strong duality)

↳ Applications of dual: • $y_1, \dots, y_m$ = shadow prices = changes of profit (per unit capacity) at current operating conditions

• Value of company (in terms of operational profit) = all ressources valued at shadow prices

2.5 Transportation Problems

Very important for logistics!

↳ An LP problem with very specific constraints:

Minimize $Z = \sum_{ij} c_{ij} x_{ij}$,

subject to $\sum_j x_{ij} = s_i$ (supply rule), $\sum_i x_{ij} = d_j$ (demand rule), and $x_{ij} \geq 0$.

- If $\sum_i s_i = \sum_j d_j$ (supply = demand) then there feasible sol.
- Integer property

Study how to deal with extra difficulties using dummy sources/sinks/variables.

2.6 Network Optimization

Ubiqitous in applications!

Problem types: • shortest path
- minimum spanning tree
- maximum flow

Study the algorithms to solve these problem types by hand

Overarching framework: Minimum cost flow problem:

Minimize $Z = \sum_{i,j} c_{ij} x_{ij}$

subject to $\underbrace{\sum_j x_{ij}}_{\text{outflow}} - \underbrace{\sum_j x_{ji}}_{\text{inflow}} = b_i$, and $0 \le x_{ij} \le u_{ij}$ (capacity constraint)

Extra example: project management (critical path, activity crashing)

3. Further Optimization Techniques

From here on, we go beyond LP framework

3.1 Dynamic Programming

↳ Very useful general strategy for many applications
↳ General principle: divide problem into different stages; in each state an optimal policy decision needs to be made

Solution technique:

- We start at the end, and keep track of the optimal costs
- $f_i(s, x_i)$ = cost of optimal route starting at s (stage $i-1$), going through x_i (at stage i), and optimal from then on

  $= c_{s,x_i} + \underbrace{f^*_{i+1}(x_i)}_{\text{or variations thereof, depending on problem context}}$
- $f_i^*(s) = \min_{x_i} f_i(s, x_i)$, the minimum x_i^* is the optimal policy decision (in stage i)

Study how to set this up and solve it with an $\begin{array}{c|c} & x_i \rightarrow \\ \hline s \downarrow & \end{array}$ table.

Probabilistic Dynamic Programming

↳ here: minimize <u>expected</u> costs (although a more comprehensive analysis might take other factors into account, e.g., variance)

↳ probabilities enter into $f_i(s, x_i)$ function

3.2 Decision Analysis

This is another important application:

- How to make decisions when consequences are uncertain?
- How to use $\underbrace{\text{additional information}}_{\text{experimentation}}$ (which comes at a cost) to improve decisions?
- As before, we only consider expected profit here.

Key tools: • Bayes' rule: $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B)}$

• Decision trees

3.3 Inventory Theory

Important practical problem!

We studied variations of the EOQ model:

- Basic version: setup cost K, unit cost c, holding cost h, continuous withdrawal rate d, no shortages. Then $Q^* = \sqrt{\frac{2dK}{h}}$ is the optimal order quantity.

(You don't need to memorize the formula, but you need to know how to derive it.)

- Planned shortages: penalty p for shortages. Then $Q^* = \sqrt{\frac{2dk}{h}}\sqrt{\frac{h+p}{p}}$.

Further models: • Periodic review: demand r_i in period i

↳ use dynamic programming

• Perishable products, single period, with <u>stochastic</u> demand: minimize (as before) <u>expected</u> cost

↳ Usually one uses a continuous probability distribution. Then $\Phi(y^*) = \frac{p-c}{p+h}$, where y^* = optimal order quantity, Φ = cumulative distribution. $\Phi(y^*)$ = optimal service level = probability that demand is satisfied.

↳ For exponential probability distribution we find specifically:

$$y^* = \mu \ln \frac{p+h}{c+h}$$

3.4 Nonlinear Programming

Vast field, we just mention some of the problems that can occur (and that we need to be aware of in applications!) and summarize some techniques.

Be aware what can happen if: • constraints nonlinear, objective fct. linear

• constraints linear, objective fct. nonlinear

Important practical problem (specifically for nonlinear solvers such as ipopt): local vs. global max./min.

Brief survey of different techniques so you know what types of methods are out there.