Dr. Stephan Juricke, Prof. Sören Petrat (based on lecture notes by Marcel Oliver)
Jacobs University, Fall 2022

1. Functions

1.1 Numbers and Polynomials

Topic 1.1.A: Numbers and Roots of Polynomials

Most basic type of numbers: the $\boxed{\text{natural numbers}}$ $\mathbb{N} = \underbrace{\{1,2,3,4,\dots\}}_{\text{the set containing the elements } 1,2,3,4,\dots}$.

Note: Intuitively clear what $\mathbb{N}$ is, but in Mathematics $\mathbb{N}$ can be defined by the Peano axioms.

The key operations on $\mathbb{N}$ are addition $(+)$ and multiplication $(\cdot)$. These are:

- commutative: $a+b = b+a$

  $a \cdot b = b \cdot a \qquad \underbrace{\overset{=\text{for all}}{\forall}\, a,b \overset{=\text{element of}}{\in} \mathbb{N}}_{=\text{ for all } a,b \text{ in the natural numbers}}$

- associative: $a+(b+c) = (a+b)+c$

  $a(bc) = (ab)c \qquad \forall a,b,c \in \mathbb{N}$

- distributive: $a(b+c) = ab+ac \qquad \forall a,b,c \in \mathbb{N}$

Next: We extend the set of numbers while keeping these laws true.

a) Neutral elements.

- The neutral element of multiplication is 1, since $1 \cdot a = a \quad \forall a \in \mathbb{N}$
- The neutral element of addition is $\underbrace{0}_{\notin \mathbb{N} \text{ (not an element of } \mathbb{N})}$, since $0 + a = a \quad \forall a \in \mathbb{N}$

How does 0 act in multiplication? From the distributive law we have:

$b\underbrace{(a+0)}_{=a} = ba + b0$, i.e., we have $ba = ba + b0$, i.e., $b0 = 0$.

$\Rightarrow$ We define $\boxed{\mathbb{N}_0 \underbrace{:=}_{\text{"is defined to be equal"}} \{0\} \underbrace{\cup}_{\text{union of sets}} \mathbb{N}} = \{0, 1, 2, 3, \ldots\}$

b) Solving Equations / Inverse Operations.

- Consider $a + x = b$ $(*)$ for given $a, b \in \mathbb{N}_0$. It cannot be solved in $\mathbb{N}_0$ if $a > b$.

  $\Rightarrow$ Need to introduce negative numbers: for $a \in \mathbb{N}$ we define $-a$ as the object that satisfies $-a + a = 0$.

  Then we can solve $(*)$: $x = b + (-a) = b - a$.

  $\Rightarrow$ We define the $\boxed{\text{integers } \mathbb{Z} := -\mathbb{N} \cup \{0\} \cup \mathbb{N}}$.

- Consider $ax = b$ for $a, b \in \mathbb{Z}$. It cannot be solved in $\mathbb{Z}$ in general, e.g., $4x = 3$.

  $\Rightarrow$ Define multiplicative inverse: for $a \in \underbrace{\mathbb{Z} \setminus \{0\}}_{\mathbb{Z} \text{ without } 0}$, we define $\frac{1}{a}$ as the object that satisfies $a \cdot \frac{1}{a} = 1$.

  We write $\frac{a}{b} = a \cdot \frac{1}{b}$.

  $\Rightarrow$ We define the $\boxed{\text{rational numbers } \mathbb{Q} := \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\} \right\}}$

c) Roots of Polynomials.

Definitions: • We call $x^n = \underbrace{x \cdot \ldots \cdot x}_{n \text{ times}}$ the n-th power of x.

• We call $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = \overbrace{\sum_{j=0}^{n}}^{\text{sum over } j \text{ from } 0 \text{ to } n} a_j x^j$ a polynomial. Here, $a_0, a_1, \ldots, a_n$ are given numbers (coefficients), and x is a variable (or argument).

We call n (the highest power of x with nonzero coefficient) the degree of the polynomial.

Example: $p(x) = x^2 - 2x + 3$ $(a_2 = 1, a_1 = -2, a_0 = 3)$ is a polynomial of degree 2.

Definition: The roots of a polynomial are those values of x where $p(x) = 0$.

For degree n = 1: $\underbrace{x + a_0 = 0}_{\text{here, we assume we have already divided by the coefficient } a_n} \Rightarrow x = -a_0$.

For degree n = 2: $x^2 + a_1 x + a_0 = 0$ (quadratic equation)

$$\Rightarrow \underbrace{\left(x + \frac{a_1}{2}\right)^2}_{= x^2 + a_1 x + \left(\frac{a_1}{2}\right)^2} - \left(\frac{a_1}{2}\right)^2 + a_0 = 0$$

$$\Rightarrow \left(x + \frac{a_1}{2}\right)^2 = \left(\frac{a_1}{2}\right)^2 - a_0$$

If $x^2 = a$, then $x = \sqrt{a}$ or $x = -\sqrt{a}$.

$$\Rightarrow x + \frac{a_1}{2} = \pm \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0}$$

$$\Rightarrow \boxed{x = -\frac{a_1}{2} \pm \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0}}$$

Note: For $a_2x^2 + a_1x + a_0 = 0$ we find $x = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_2}$.

We call $\Delta = a_1^2 - 4a_0a_2$ the discriminant.

Issues: • If $\Delta < 0$, then there is no solution in $\mathbb{Q}$. (Need complex numbers; next time.)

• Even if $\Delta > 0$, there might not be a solution in $\mathbb{Q}$.

Example: $x^2 = 2 \Rightarrow x = \pm\sqrt{2}$. We claim that $\sqrt{2} \notin \mathbb{Q}$.

Proof: By contradiction: Suppose $\sqrt{2} = \frac{n}{m}$ with $n, m \in \mathbb{N}$.

Then $m\sqrt{2} = n$

$$\Rightarrow m^2 \sqrt{2}^2 = n^2$$

$$\Rightarrow \underbrace{m^2 2}_{\text{Odd number of prime factors.}} = n^2$$

All prime factors of n appear an even number of times.

$\Rightarrow$ Contradiction. $\underbrace{\square}_{=\text{proof finished}}$

Numbers such as $\sqrt{2}$ are called $\overbrace{\text{irrational numbers}}^{\text{sometimes abbreviated } \mathbb{I}}$. They have an infinite and not eventually periodic decimal expansion (e.g., $\sqrt{2} = 1.41421\ldots$), while rational numbers either have a finite (e.g., $\frac{3}{20} = 0.15$) or eventually periodic (e.g., $\frac{123}{900} = 0.13999\ldots = 0.13\overline{9}$) decimal expansion.

Irrational numbers "fill in the gaps in the rational numbers".

We call the set of all rational and irrational numbers the $\boxed{\text{real numbers } \mathbb{R}}$

(also called "number continuum").

Note: The real numbers are qualitatively different from the rational numbers, and their construction is non-trivial. (See Analysis I.)

E.g., one important difference:

- $\mathbb{Q}$ is countable:

$$\begin{array}{ccccc} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \\ \frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \\ \frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \\ \vdots & & & & \ddots \end{array}$$

- $\mathbb{R}$ is not countable (There are "more" than countably infinite real numbers.)