Calculus and Elements of Linear Algebra I Session 10

Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

Jacobs University, Fall 2022

2. Derivatives

2.1 Introduction to derivatives and their properties

Topic 2.1.C: Implicit differentiation & second derivative

Implicit differentiation

For equations that are difficult to write in terms of a function, one can use implicit differentiation

Example: $x^2 + y^2 = 1$ (unit circle)

"dependent variable"

Goal: Find $y'(x) = \frac{dy}{dx}$

"independent variable"

Naively: $y = \pm\sqrt{1 - x^2} = \pm(1 - x^2)^{\frac{1}{2}}$

$$\curvearrowright \quad \frac{dy}{dx} = \underbrace{\pm \frac{1}{2}\left(1 - x^2\right)^{-\frac{1}{2}}}_{\text{outer}} \cdot \underbrace{\left(-2x\right)}_{\text{inner}}$$

$$= \frac{-x}{\underbrace{\pm\sqrt{1 - x^2}}_{= y}} = -\frac{x}{y}$$

We get the same via implicit differentiation :

Suppose $y = y(x)$ and differentiate $x^2 + y^2 = 1$

w.r.t. x : $\boxed{\qquad\qquad\qquad\qquad}$ $\Rightarrow$

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Second derivative

If we differentiate the derivative again, we get the second derivative, i.e. the local change of the slope of the function.

Ex.: $f(x) = x^2$, $f'(x) =$, $f''(x) =$

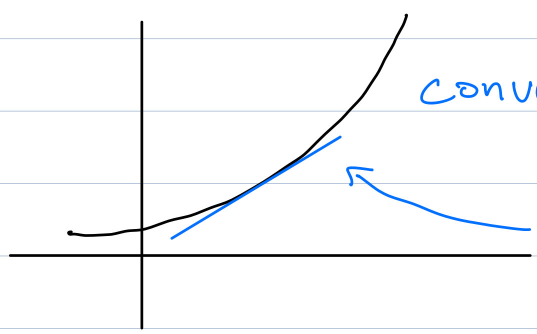
We can also write

$$f''(x) = \frac{d^2 f}{dx^2} = \ddot{f}$$

If the change of the slope of the function is positive at a point, the slope is getting larger, i.e. the curve is curving upwards:

$$f''(x) > 0$$

f is concave up or convex



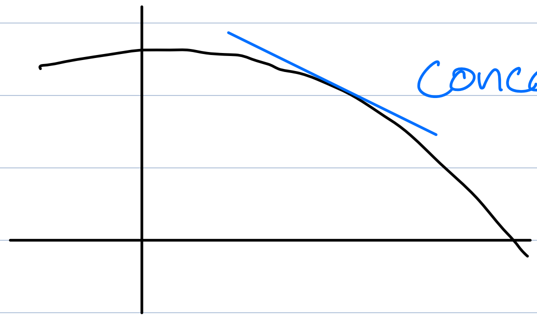
convex, i.e. curving upwards or has positive curvature

(tangent line is below graph)

If the change of the slope of the function is negative at a point, the slope is getting smaller, i.e. the curve is curving downwards:

$$f''(x) < 0$$

f is concave down or concave



concave, i.e. curving downwards
or has negative curvature
(tangent line is above graph)

If $\boxed{f''(x) = 0, \text{ the slope is not changing}}$ at x, i.e. the slope remains constant at x and the curve does not curve up or down at x.

Example of entirely convex function:
$f(x) = x^2$

Example of entirely concave function:
$f(x) = -x^2$

Furthermore:

If $f'' > 0$, then f' is increasing

If $f'' < 0$, then f' is decreasing

So if f'' changes sign at a point x, then the rate of change of the rate of change (in physical terms the acceleration) changes sign.

Such a point is called **point of inflection** ($f''(x) = 0$)