Calculus and Elements of Linear Algebra I Session 11

Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

Jacobs University, Fall 2022

2. Derivatives

2.1 Introduction to derivatives and their properties

Topic 2.1.D: Theorems of differentiation

Theorem: $f : (a,b) \to \mathbb{R}$ diff'able at $x \in (a,b)$

$$\Rightarrow f \text{ cont. at } x$$

Proof: $\lim\limits_{h \to 0} \big(f(x+h) - f(x)\big) = \lim\limits_{h \to 0} h \cdot \frac{f(x+h) - f(x)}{h}$

$$= \underbrace{\lim_{h \to 0} h}_{=0} \cdot \underbrace{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{= f'(x)} = 0$$

since f diff'able at x

$$\Rightarrow \quad \lim_{h \to 0} f(x+h) = f(x) \quad \blacksquare$$

(with $\lim_{h \to 0} f(x) = f(x)$ as $f(x)$ independent of h)

Reminder: f cont. at x_0: $\lim_{x \to x_0} f(x) = f(x_0)$

$$\Leftrightarrow \lim_{x \to x_0} f(x) - f(x_0) = 0 \Leftrightarrow \lim_{h \to 0} f(x_0+h) - f(x_0) = 0$$

<u>Note:</u> As mentioned before, f continuous (in general) <u>does not</u> imply f diff'able:

<u>e.g.:</u> $f(x) = |x|$ at $x = 0$

(sharp change of function)

Diff'able functions are even "smoother" than continuous functions.

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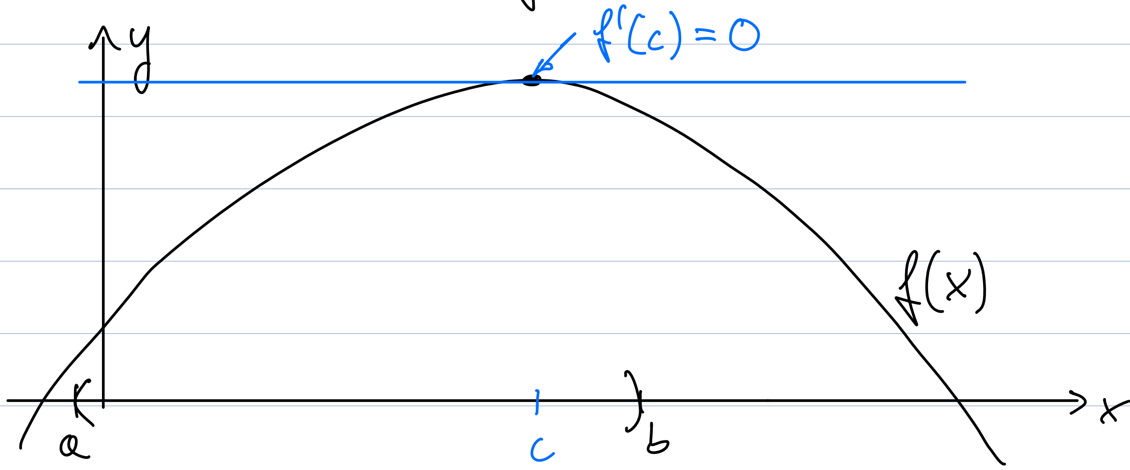
Theorem: $f : (a,b) \to \mathbb{R}$ diff'able

Suppose f has a maximum (or minimum) at some $c \in (a,b)$.

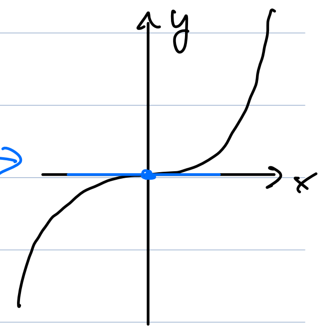
Then: $f'(c) = 0$



Note: $f'(x)=0$ does not imply that f has a max (or min) at x!

Eg.: $f(x)=x^3$, $f'(x)=3x^2$

$\Rightarrow f'(0)=0$



yet f is increasing and does not have a max at $x=0$.

Proof of theorem: If the max of f in (a,b) is taken at c, then $f(c+h)-f(c)\leq 0$

take $h \geq 0$	take $h<0$
$\dfrac{\overbrace{f(c+h)-f(c)}^{\leq 0}}{\underbrace{h}_{>0}} \leq 0$	$\dfrac{\overbrace{f(c+h)-f(c)}^{\leq 0}}{\underbrace{h}_{<0}} \geq 0$
$\Rightarrow \underbrace{\lim_{h\searrow 0} \dfrac{f(c+h)-f(c)}{h}}_{=f'(c)} \leq 0$	$\Rightarrow \underbrace{\lim_{h\nearrow 0} \dfrac{f(c+h)-f(c)}{h}}_{=f'(c)} \geq 0$

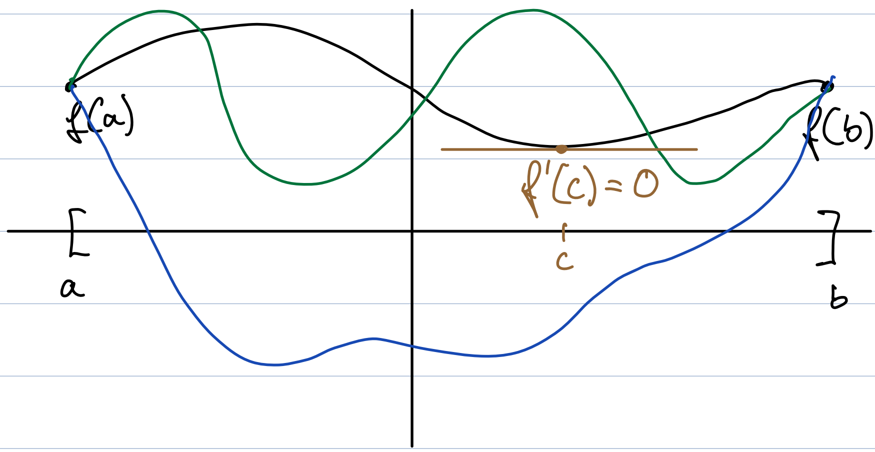
as f is diff'able

$\Rightarrow f'(c)=0$ $\blacksquare$

Rolle's theorem

$f : [a,b] \longrightarrow \mathbb{R}$ cont., diff'able on (a,b), $f(a) = f(b)$

$\Rightarrow \exists\, c \in (a,b)$ such that $f'(c) = 0$



In words: A "nice" function f has a root of f' between any two roots of f (i.e. if $f(a)=f(b)=0$ $\Rightarrow \exists\, c :\ f'(c)=0$)

<u>Proof:</u> If f is a constant, then the statement is true because $f'=0$ on (a,b).

Otherwise, as f is continuous, it must have its min and max on $[a,b]$

and at least one of them lies inside of (a,b) (and not at a or b, as $f(a) = f(b)$, which cannot be both min & max).

Now we apply the previous theorem. ∎

<u>Def.:</u> We say $f(x)$ has a critical point at x if $f'(x) = 0$

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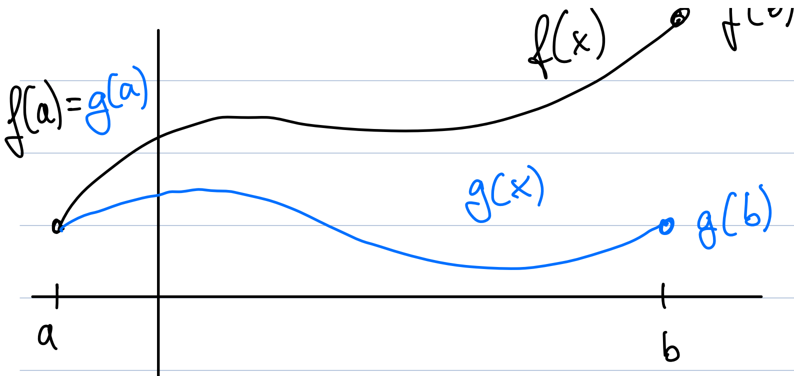
$f: [a,b] \longrightarrow \mathbb{R}$ cont., diff'able on (a,b), $f(a) = f(b)$

$$\Longrightarrow \exists\, c \in (a,b) \text{ such that } f'(c) = 0$$

What if $f(a) \neq f(b)$

We have to use a different function.



$$g(x) = f(x) - \underbrace{\frac{x-a}{b-a}}\big(f(b) - f(a)\big)$$

$x = a \Rightarrow \frac{x-a}{b-a} = 0$

$x = b \Rightarrow \frac{x-a}{b-a} = 1$

$\Rightarrow \quad g(a) = f(a), \quad g(b) = f(b) - \big(f(b) - f(a)\big) = f(a)$

$\Rightarrow \quad g(a) = g(b)$

Take derivative: $\quad g'(x) = f'(x) - \frac{f(b) - f(a)}{b-a}$

Rolle's theorem: $\exists\, c \in (a,b)$ s.t. $g'(c) = 0$

$$\Rightarrow \quad f'(c) = \frac{f(b) - f(a)}{b-a}$$

instantaneous rate of change

average rate of change on interval (a,b)

$\Rightarrow$

Mean value theorem (MVT)

$$f : [a,b] \to \mathbb{R} \text{ cont., diff'able on } (a,b) \Rightarrow \exists\, c \in (a,b) \text{ s.t. } \frac{f(b)-f(a)}{b-a} = f'(c)$$

In words: If f is "nice", the average rate of change over an interval equals the instantaneous rate of change somewhere within the interval.

Ex.: $f(x) = \ln x$, $f'(x) = \frac{1}{x}$

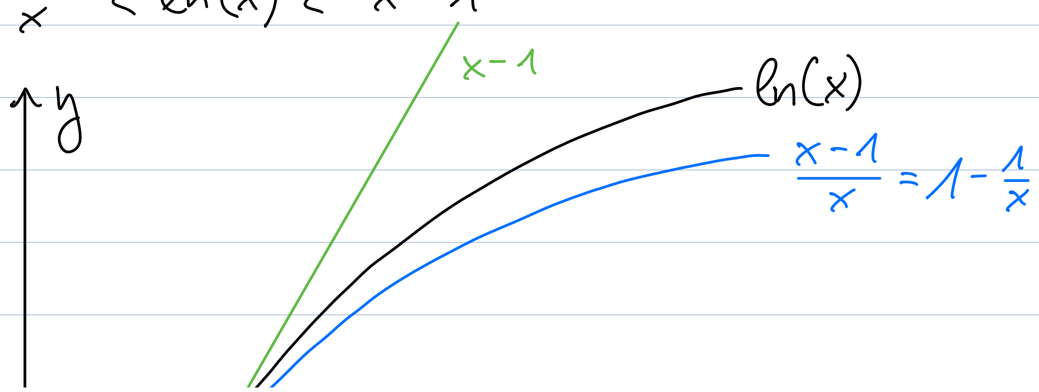
$$\frac{\ln(x) - \underbrace{\ln(1)}_{=0}}{x-1} = \frac{1}{c} \quad \text{for } c \in (1,x),\ x>1$$

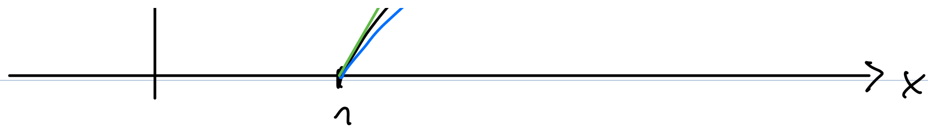
$$\Rightarrow \ln(x) = \frac{x-1}{c}$$

Since $c>1$, it follows for $c=1$: $\ln(x) < x-1$

Since $c<x$, it follows for $c=x$: $\ln(x) > \frac{x-1}{x}$

$$\Rightarrow \quad \frac{x-1}{x} < \ln(x) < x-1$$





Important consequence of MVT

$f:(a,b)\to\mathbb{R}$ diff'able with $f'(x)\geq 0$ (≤ 0)

$\Rightarrow f$ is increasing (decreasing) on (a,b)

i.e. whenever $x_1 < x_2$; $x_1, x_2 \in (a,b)$
we have $f(x_1)\leq f(x_2)$ $\quad (f(x_1)\geq f(x_2))$

This can be adjusted to the case when $f'(x)>0$ (<0).
$\Rightarrow f$ is strictly increasing (decreasing)
(although it is not actually necessary for $f'(x)>0$ everywhere; if $f'(x)=0$ at some distinct points, f is still strictly increasing)

<u>Ex.:</u> $f(x)=x^3+x-1$, $f'(x)=3x^2+1>0 \quad \forall x\in\mathbb{R}$

$\Rightarrow f$ is strictly increasing as $f'(x)>0$;
it can therefore have at most one real root.

Since $\lim\limits_{x\to\infty} f(x)=\infty$, $\lim\limits_{x\to-\infty} f(x)=-\infty$, it has exactly one root.