Calculus and Elements of Linear Algebra I — Session 12

Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

Jacobs University, Fall 2022

# 2. Derivatives

## 2.2 Applications of differentiation

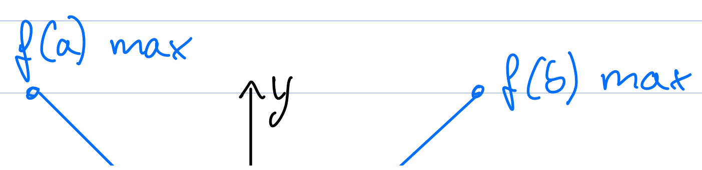
### Topic 2.2.A: Extreme value problems

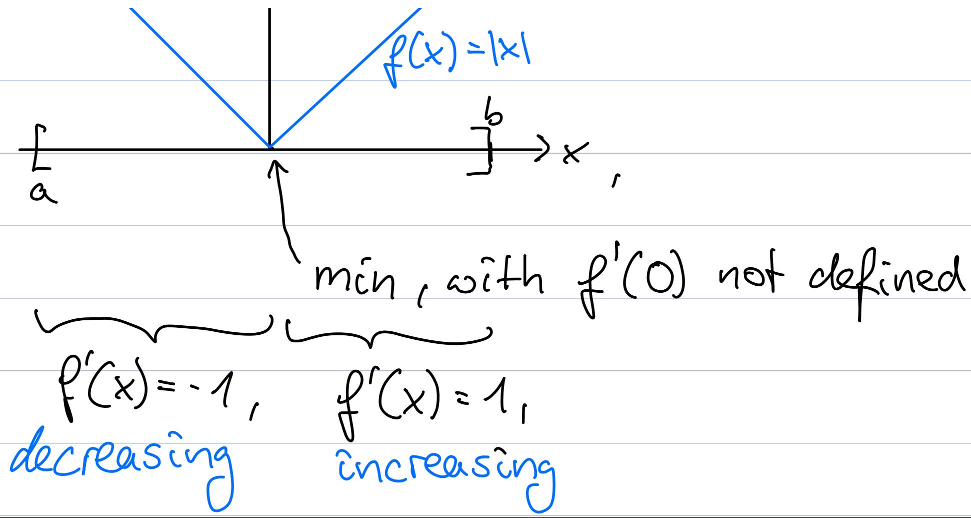
#### Conditions for extreme values

Possible extrema:

- Critical points
- Points where $f'(x)$ is not defined
- End points of closed intervals of definition

Eg.:



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<u>Sufficient conditions</u> for existence of extreme values:

- If $f'$ changes sign at the point in question
- $f''(x)$ exists and is non-zero:

  → if $f''(x) > 0$, then (provided $f''$ exists and is continuous near $x$) $f'$ is increasing. Hence, if $f'(x) = 0$, it must change sign from $-$ to $+$, so $f$ has a min at $x$

  → if $f''(x) < 0$, then (provided $f''$ exists and is continuous near $x$) $f'$ is decreasing. Hence, if $f'(x) = 0$, it must change sign from $+$ to $-$, so $f$ has a max at $x$

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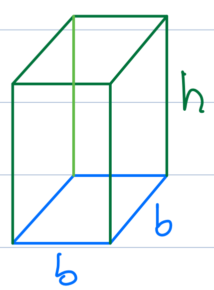
Example:
Construct a rectangular container with square base:

- material for base $5 \frac{€}{m^2}$
- material for sides/top $1 \frac{€}{m^2}$




What is the largest possible volume for 72€?

Volume $V = b^2 \cdot h$ (base length $b$ · height $h$)

costs $c = 5 \cdot b^2 + 1 \cdot (b^2 + 4bh)$ (in €)

base area   top area   4 sides area

cost c should be 72 €

$\Rightarrow \quad 72 = 5b^2 + b^2 + 4bh \Rightarrow \cancel{6}^{3} b^2 + \cancel{4}^{2} bh = \cancel{72}^{36}$

Solve for h: $\quad h = \dfrac{36 - 3b^2}{2b} = 18\,\dfrac{1}{b} - \dfrac{3}{2}\,b$

$\Rightarrow \quad V(b) = b^2 \cdot \underbrace{\left(18\,\dfrac{1}{b} - \dfrac{3}{2}\,b\right)}_{=h} = 18b - \dfrac{3}{2}\,b^3$

$\Rightarrow \quad V'(b) = 18 - \dfrac{3}{2} \cdot 3b^2 = 18 - \dfrac{9}{2}\,b^2 \overset{!}{=} 0$

to get extreme value

$\Rightarrow \quad 4 = b^2 \Rightarrow \underline{b = 2}$ (negative b does not make sense)

$\Rightarrow \quad V''(b) = -9b \underline{< 0}$ for $b > 0$, so we have a max. at $b = 2$ (in m)

$\Rightarrow \quad h = 18\,\dfrac{1}{2} - \dfrac{3}{2} \cdot 2 = 9 - 3 = \underline{6}$ (in m)

$\Rightarrow \quad \boxed{V = 2^2 \cdot 6 = 24}$ (in $m^3$)

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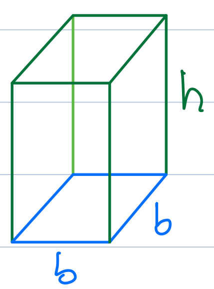


What is the largest possible volume for 72 € ?

Alternative solution using implicit differentiation

Suppose both h and b are functions of some artificial parameter $t$ (i.e. $h(t)$ and $b(t)$).

Then we have two conditions for the two unknowns $h$ and $b$:

- $\frac{dV}{dt} \overset{!}{=} 0$ (necessary condition for max. volume) with $\boxed{V = b^2 \cdot h}$ ⊗
- $\boxed{3b^2 + 2bh = \text{constant}}$ ⊛ (total cost in €, here 72)

We use implicit differentiation:

$$\frac{dV}{dt} = 2b\,\frac{db}{dt}\cdot h + b^2\,\frac{dh}{dt} \overset{!}{=} 0$$ (Product and chain rules)

$$\Rightarrow \boxed{2bh\,\frac{db}{dt} = -b^2\,\frac{dh}{dt}} \quad 1$$

and differentiating ⊛ on both sides gives:

$$3\cdot 2b\,\frac{db}{dt} + 2h\,\frac{db}{dt} + 2b\,\frac{dh}{dt} = 0$$

$$\Rightarrow \boxed{(6b+2h)\,\frac{db}{dt} = -2b\,\frac{dh}{dt}} \quad 2$$

Divide 1 by 2:
(to get rid of $\frac{db}{dt}$ & $\frac{dh}{dt}$)

$$\frac{\cancel{2}\cancel{b}h\,\cancel{\frac{db}{dt}}}{(\cancel{6}^{3}b+\cancel{2}h)\cancel{\frac{db}{dt}}} = \frac{-\cancel{b}^{\cancel{2}}\,\cancel{\frac{dh}{dt}}}{-2\cancel{b}\,\cancel{\frac{dh}{dt}}}$$

$$\Rightarrow \frac{h}{3b+h} = \frac{1}{2} \Rightarrow 2h = 3b+h \Rightarrow \boxed{h = 3b}$$

Now we can get other quantities ($b$ and finally $V$) by inserting into $\circledast$ and $\otimes$.

This approach avoids solving before differentiating, so it works more generally.

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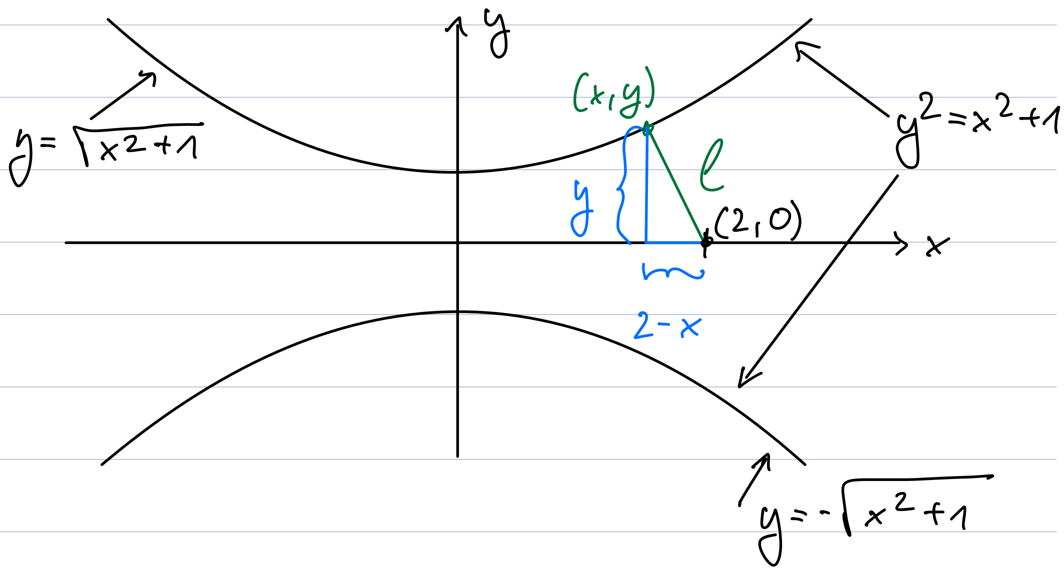
Example: What is the min. distance of point $(2,0)$ to graph of $y^2 = x^2 + 1$?

Sketch:



$\ell^2 = (2-x)^2 + y^2$

Here: $\ell^2 = (2-x)^2 + x^2 + 1$ as $y^2 = x^2+1$

Note: Minimizing $\ell$ means also minimizing $\ell^2$!

$\Rightarrow \quad 0 \overset{!}{=} \frac{d\ell^2}{dx} = \underbrace{2(2-x)}_{\text{outer}} \cdot \underbrace{(-1)}_{\text{inner}} + 2x = 4x - 4$

$\Rightarrow \quad \boxed{x = 1} \qquad \Rightarrow \quad \boxed{y = \sqrt{2}}$ and $\boxed{\ell = \sqrt{3}}$

Why is this a min.? We could look at second derivative, but its easier here:

$\lim_{x \to \pm\infty} \ell^2 = \infty$ and only one

critical point at $x = 1$

$\Rightarrow$ $\ell^2$ cannot have a max., so critical point must correspond to min.

(You can also see from sketch that there cannot be a max. $< \infty$)