Calculus and Elements of Linear Algebra I Session 13

Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

Jacobs University, Fall 2022

2. Derivatives

2.2 Applications of differentiation

Topic 2.2.B: Graph sketching

Goal: Find as many qualitative features of the graph of a function as possible using Calculus, applying the rules and methods we have covered so far.

Checklist:

1. Domain of f : $D(f)$
2. Intercepts (y-, x-intercept if possible)
3. Horizontal asymptotes ($\lim_{x \to \pm\infty} f(x)$)
4. Vertical asymptotes
   $\lim_{x \to x_0} f(x)$ if x_0 is a boundary point of $D(f)$

($\lim_{x \searrow x_0} f(x)$ and $\lim_{x \nearrow x_0} f(x)$ if possible and not the same)

5. First derivative: Critical points, intervals where f is in-/decreasing
6. Second derivative: Points of inflection, intervals where f is concave up/down

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Example: $f(x) = x^4 - 6x^3 = x^3(x-6)$

1. $\mathcal{D}(f) = \mathbb{R}$ (all real numbers)

2. y - intercept: set $x = 0$

$\Rightarrow f(0) = 0 \cdot (0-6) = 0$

so $x = y = 0$ is both x- and y- intercept

another x- intercept: set $y = 0$

$\Rightarrow 0 = x^3(x-6)$

$\Rightarrow \quad y = 0 \quad$ if $\quad x = 0 \quad$ or $\quad (x-6) = 0 \Rightarrow x = 6$

3. $\lim\limits_{x \to \infty} f(x) = \infty \quad$ and $\quad \lim\limits_{x \to -\infty} f(x) = \infty$

(as $x^3 \to \infty$ and $x - 6 \to \infty$ for $x \to \infty$) $\quad$ (as $x^3 \to -\infty$ and $x - 6 \to -\infty$ for $x \to -\infty$)

4. No vertical asymptotes because $D(f) = \mathbb{R}$

5. $f'(x) = 4x^3 - 6 \cdot 3x^2 = \underbrace{2x^2}_{>0 \text{ for } x \neq 0}(2x - 9)$

need to check sign of $2x - 9$

$\Rightarrow \quad f'(x) \leq 0 \quad$ for $\quad 2x - 9 \leq 0$, so $2x \leq 9$ or $x \leq \frac{9}{2}$

$f'(x) > 0 \quad$ for $\quad 2x - 9 > 0$, so $2x > 9$ or $x > \frac{9}{2}$

$\Rightarrow \quad f$ has min. at $\quad x = \frac{9}{2}$, $y = f\left(\frac{9}{2}\right) = -136.6875$ with $f'\left(\frac{9}{2}\right) = 0$

Note: Another critical point is at $x = 0$ as $2x^2 = 0$ for $x = 0$.
But it is neither min. nor max. as

$f'(x)$ does not switch sign there.

6. $f''(x) = \left(4x^3 - 18x^2\right)' = 12x^2 - 36x$
$= 12x(x-3)$

Find x for which $f''(x) = 0$ and then check signs of factors around these x values.

$f''(x) \overset{!}{=} 0 \overset{!}{=} 12x(x-3)$, so for $x=0$ and $x=3$

Check sign: If $x < 0$, $12x < 0$ & $x-3 < 0$
$\Rightarrow f''(x) > 0$
$\Rightarrow f$ is concave up or convex

If $x \in (0,3)$, $12x > 0$ & $x-3 < 0$
$\Rightarrow f''(x) < 0$
$\Rightarrow f$ is concave down or just concave

If $x > 3$, $12x > 0$ & $x-3 > 0$
$\Rightarrow f''(x) > 0$
$\Rightarrow f$ is concave up or conex

$\Rightarrow$ points of inflection at $x = 0$ $(y = f(0) = 0)$ and

$x = 3$ $(y = f(3) = -81)$

Sketch: Different scale for x- and y-axis!