Calculus and Elements of Linear Algebra I — Session 14

Dr. Stephan Juricke, Prof. Sören Petrat (based on lecture notes by Marcel Oliver)
Jacobs University, Fall 2022

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# 3. Integrals

## Topic 3.A: Indefinite Integrals

"Integration = opposite to differentiation"

sometimes called "primitive"

**Definition:**
Let $f, F: I \to \mathbb{R}$ ($I$ some interval), $F$ differentiable. Then $F$ is an antiderivative of $f$ if $F' = f$.

Example: $f(x) = x^2 \Rightarrow F(x) = \frac{1}{3}x^3$ is an antiderivative, because $F'(x) = x^2$.
But note: $G(x) = \frac{1}{3}x^3 + 4$ is also an antiderivative.

We have:

**Theorem:**
$F$ and $G$ are antiderivatives of $f$ if and only if $G = F + c$ for some $c \in \mathbb{R}$.

**Proof:**
"$\Leftarrow$" If $G = F + c$, then $G' = (F+c)' = F' + \underbrace{c'}_{=0} = F'$, i.e., $F$ and $G$ are antiderivatives of the same function.

"$\Rightarrow$" Assume $F' = f$ and $G' = f$. Define $H = G - F$; then $H' = G' - F' = f - f = 0$.
By the mean-value theorem: $\frac{H(y) - H(x)}{y - x} = \underbrace{H'(z)}_{=0}$ for some $z \in (x, y)$.

$\Rightarrow H(y) = H(x)$ for $x \neq y$, i.e., $H(x) = c$ for some $c \in \mathbb{R}$. $\square$

**Definition:**

We call $\underbrace{\int f(x)\,dx = F(x) + c}$ the indefinite integral. Here, $c \in \mathbb{R}$ is called "constant of integration."

here, $F$ is an antiderivative of $f$

Examples:

- $\frac{d}{dx} x^{n+1} = (n+1)x^n$ for $n \in \mathbb{R}, n \neq -1$ $\Rightarrow$ $\boxed{\int x^n\,dx = \frac{1}{n+1} x^{n+1} + c, \ n \neq -1.}$

- $\frac{d}{dx} \ln x = \frac{1}{x}$ for $x > 0$ $\Rightarrow$ $\boxed{\int \frac{1}{x}\,dx = \ln x + c, \ x > 0.}$

  Note: for $x < 0$ we have $\frac{d}{dx} \ln(-x) = -\frac{1}{(-x)} = \frac{1}{x}$, so $\int \frac{1}{x}\,dx = \ln|x| + c$ on any interval not including 0.

- $\frac{d}{dx} \cos x = -\sin x$, $\frac{d}{dx} \sin x = \cos x$ $\Rightarrow$ $\boxed{\int \sin x\,dx = -\cos x + c \text{ and } \int \cos x\,dx = \sin x + c.}$

- $\frac{d}{dx} e^x = e^x$ $\Rightarrow$ $\boxed{\int e^x\,dx = e^x + c.}$

- $\frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$

  $\Rightarrow$ $\boxed{\int \frac{1}{\cos^2 x}\,dx = \tan x + c.}$

- $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$ $\Rightarrow$ $\boxed{\int \frac{1}{1+x^2}\,dx = \arctan x + c.}$

  recall derivative of inverse function

Next: rules of integration which follow from key rules of differentiation.

- Product rule: $\frac{d}{dx}(F(x)g(x)) = F'(x)\,g(x) + F(x)\,g'(x)$

$$\Rightarrow F(x)g(x) = \int F'(x)g(x)\,dx + \int F(x)g'(x)\,dx$$

This is called $\boxed{\text{"integration by parts": } \int F'(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)\,dx.}$

Helpful when we know antiderivative of one fct. ($F'$) and when it helps to take the derivative of the other ($g$).

Example: $\int \underbrace{e^x}_{F'} \underbrace{x}_{g}\,dx = \underbrace{e^x}_{F} \underbrace{x}_{g} - \int \underbrace{e^x}_{F} \underbrace{1}_{g'}\,dx = e^x x - e^x + c = e^x(x-1) + c.$

- Chain rule: $\frac{d}{dx} F(u(x)) = F'(u(x))\,u'(x) = f(u(x))\,u'(x)$ $\quad$ ($F' = f$ here).

This leads to $\boxed{\text{"integration by substitution": } \int f(u(x))\,u'(x)\,dx = F(u(x)) + c.}$

Useful for memorization (not rigorous): write $u'(x) = \frac{du}{dx} \Rightarrow u'(x)\,dx = du$

$$\Rightarrow \int f(u(x))\,u'(x)\,dx = \int f(u)\,du = F(u(x)) + c.$$

Examples:

- $\int \underbrace{e^x}_{u'(x)} \underbrace{\cos(e^x)}_{f(u(x))}\,dx = \int \cos(u)\,du = \sin(u(x)) + c = \sin(e^x) + c.$

$(u(x) = e^x,\ u'(x) = e^x,\ f(y) = \cos(y))$

- $\int \underbrace{\sqrt{1+x^3}}_{f(u(x))} \underset{\frac{1}{3}u'(x)}{x^2}\,dx = \int f(u(x))\,\frac{1}{3}u'(x)\,dx = \frac{1}{3}\int f(u)\,du = \frac{1}{3}\int \sqrt{u}\,du = \frac{1}{3}\,\frac{2}{3}\,u(x)^{\frac{3}{2}} + c$

$= \frac{2}{9}\left(1+x^3\right)^{\frac{3}{2}} + c.$

$\left(u(x) = 1+x^3,\ u'(x) = 3x^2,\ f(y) = \sqrt{y}\right)$