Calculus and Elements of Linear Algebra I Session 15

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3. Integrals

Topic 3.B: Integration of Rational Functions

Definition:

A rational function $r(x)$ is a ratio of two polynomials $P(x)$ and $Q(x)$, i.e., $r(x) = \frac{P(x)}{Q(x)}$.

We already know how to integrate some rational functions:

- $\int \frac{1}{1+x^2}\,dx = \arctan x + c$
- $\int \frac{1}{x}\,dx = \ln|x| + c$

What about, e.g., $r(x) = \frac{x^3+x}{x^2-1}$? $\int r(x)\,dx = ?$

Note: $(x^3+x) : (x^2-1) = x + \frac{2x}{x^2-1}$ by polynomial long division.

$$\begin{array}{l} -\,(x^3 - x) \\ \hline \quad\quad 2x \end{array}$$

$$\Rightarrow \int \frac{x^3+x}{x^2-1}\,dx = \int x\,dx + \underbrace{\int \frac{2x}{x^2-1}\,dx}_{} = \frac{x^2}{2} + \ln|x^2-1| + c.$$

substitution: $= \int \frac{1}{u}\,du = \ln|u| + c = \ln|x^2-1| + c$

$u(x) = x^2-1$

$u'(x) = 2x$

$f(y) = \frac{1}{y}$

This last example can be generalized.

In general: A rational function $r(x) = \frac{P(x)}{Q(x)}$ can be written as a linear combination (= sum) of the following terms:

(i) A polynomial of degree $\deg P - \deg Q$,

(ii) rational functions of the form $\frac{A_1}{x-x_i}, \frac{A_2}{(x-x_i)^2}, \dots, \frac{A_k}{(x-x_i)^k}$, where x_i is a root of $Q(x)$ of multiplicity k,

(iii) rational functions of the form $\frac{A_1 + B_1 x}{ax^2+bx+c}, \dots, \frac{A_m + B_m x}{(ax^2+bx+c)^m}$, when $(ax^2+bx+c)^m$ is a factor of $Q(x)$.

Why?
- Do polynomial long division until a rational function $\frac{\tilde{P}}{Q}$ with $\deg \tilde{P} < \deg Q$ is left over. $\longrightarrow$ (i)
- Decompose $\frac{\tilde{P}}{Q}$ into "partial fractions". First, find roots of Q. Then we have
  - terms of the form (ii) for real roots,
  - for complex roots:
    - either terms of the form (ii), but then there are complex numbers in denominator,
    - or terms of the form (iii).

Example: $r(x) = \dfrac{4x^3+23x^2+45x+27}{x^3+5x^2+8x+4}$

First: $(4x^3+23x^2+45x+27) : (x^3+5x^2+8x+4) = 4 + \dfrac{3x^2+13x+11}{x^3+5x^2+8x+4}$ $\begin{array}{l} \leftarrow \tilde{P}(x) \\ \leftarrow Q(x) \end{array}$

$$\begin{array}{r} -\ (4x^3+20x^2+32x+16) \\ \hline 3x^2+13x+11 \end{array}$$

Next: roots of $Q(x)$: -1 is a root by guessing: $(-1)^3+5+8(-1)+4=0$.

$$(x^3+5x^2+8x+4):(x+1)=x^2+4x+4=(x+2)^2$$
$$\begin{array}{l} -\underline{(x^3+x^2)} \\ \quad 4x^2+8x \\ \quad -\underline{(4x^2+4x)} \\ \qquad 4x+4 \\ \qquad -\underline{(4x+4)} \\ \qquad\quad 0 \end{array}$$

$\Rightarrow$ 3 real roots: -1, and -2 with multiplicity $k=2$.

Thus we should expect the following decomposition into partial fractions:

$$\frac{3x^2+13x+11}{(x+1)(x+2)^2}=\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{(x+2)^2} \quad \text{for some } A,B,C.$$

We compute $\overset{\searrow}{=} \dfrac{A(x+2)^2+B(x+1)(x+2)+C(x+1)}{(x+1)(x+2)^2}$

Now we compare: numerator $=3x^2+13x+11=A(x+2)^2+B(x+1)(x+2)+C(x+1)$.

easiest to compare at the zeroes

Comparison at -1: $\quad 3-13+11=A(-1+2)^2+0+0$

$\Rightarrow A=1$

Comparison at -2: $\quad 3\cdot 4-26+11=0+0+C(-2+1)$

$\Rightarrow C=3$

easy to compare here

Comparison at 0: $\quad 11=4A+2B+C=4+2B+3$

$\Rightarrow B=2$

$$\Rightarrow \int \frac{4x^3+23x^2+45x+27}{x^3+5x^2+8x+4}\,dx=\int\left(4+\frac{1}{x+1}+\frac{2}{x+2}+\frac{3}{(x+2)^2}\right)dx$$

$$=4x+\ln|x+1|+2\ln|x+2|-\frac{3}{x+2}+C.$$