Calculus and Elements of Linear Algebra I Session 17

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Jacobs University, Fall 2022

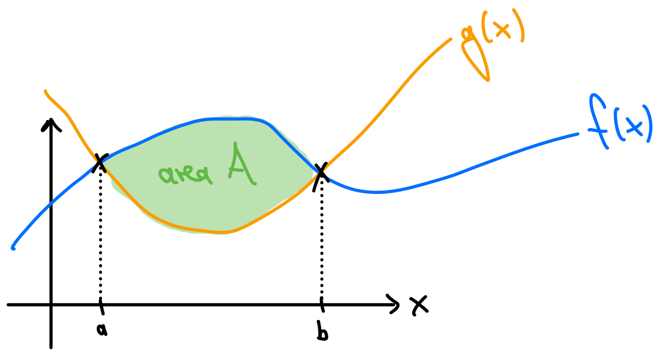
3. Integrals

Topic 3.D: Applications of Integration

Today: Some applications of (definite) integrals

Area between curves:

Compute area A between f and g:



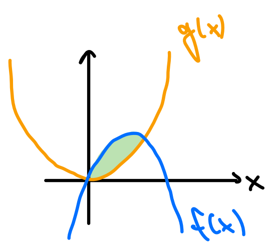
How? $\longrightarrow$ Find points of intersection.

$\longrightarrow$ Use integration to compute $\boxed{\int_a^b (f(x)-g(x))\,dx}$.

Examples:

- $g(x) = x^2$

  $f(x) = 6x - 2x^2$



Points of intersection: $f(x) = g(x) \Rightarrow x^2 = 6x - 2x^2 \Rightarrow 3x^2 - 6x = 0$

$\Rightarrow x(x-2) = 0$

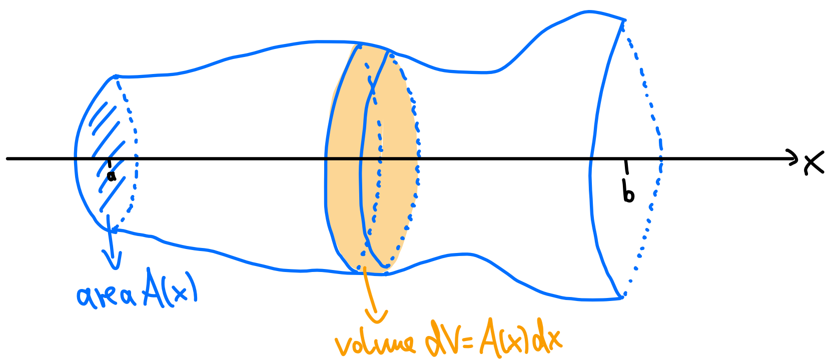
Points of intersection are at $x = 0 =: a$ and $x = 2 =: b$.

$$\Rightarrow \text{area } A = \int_0^2 (f(x)-g(x))\,dx = \int_0^2 (6x-2x^2-x^2)\,dx = \int_0^2 (-3x^2+6x)\,dx$$

$$= -x^3+3x^2 \Big|_0^2 = -8+12-0 = 4$$

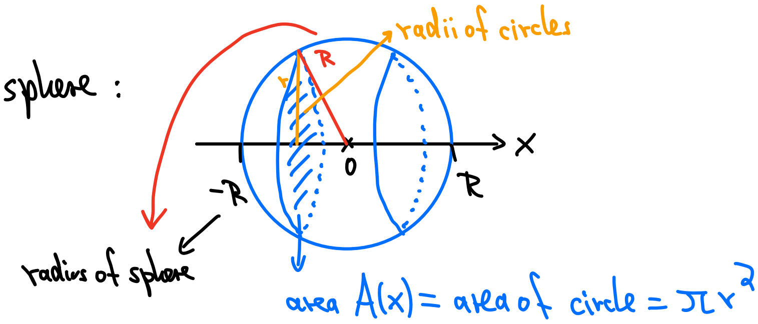
Volume computation:

Compute volume V if areas $A(x)$ of cross sections are known:



Here, volume $\boxed{V = \int_a^b \underbrace{A(x)\,dx}_{\text{"}=dV\text{"}}}$

Example: volume V of a sphere :



Note: $x^2+r^2=R^2 \Rightarrow A(x) = \pi r^2 = \pi(R^2-x^2)$

$$\Rightarrow V = \int_{-R}^{R} A(x)\,dx = \int_{-R}^{R} \pi(R^2-x^2)\,dx = \pi\left[R^2x - \tfrac{1}{3}x^3\right]_{-R}^{R}$$

$$= \pi\left[R^3 - \frac{R^3}{3} - \left(-R^3 + \tfrac{1}{3}R^3\right)\right]$$

$$= \tfrac{4}{3}\pi R^3$$

Work in physics:

Work = force × length

Moving an object from a to b by a force $F(x)$ requires work $\boxed{W=\int_a^b F(x)\,dx}$.

Example: Particle of mass m is moving along trajectory $x(t)$.

Let:
- $x(t)$ = particle position at time t
- $v(t)=\frac{dx(t)}{dt}$ = velocity
- $a(t)=\frac{dv(t)}{dt}=\frac{d^2x(t)}{dt^2}$ = acceleration

Newton's second law is $F=ma$

$$\Rightarrow \text{work } W=\int_a^b F(x)\,dx = m\int_a^b \frac{dv(t)}{dt}\,dx$$

substitution $t(x)=t$

$\Rightarrow \frac{dx}{dt}=v(t) \Rightarrow dx=v(t)\,dt$

$x(t_a)=a,\ x(t_b)=b$

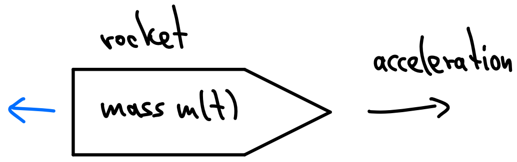
$$= m\int_{t_a}^{t_b} \frac{dv}{dt}\,v\,dt$$

$$= m\int_{t_a}^{t_b}\Big[\underbrace{\frac{d}{dt}\Big(\frac{1}{2}v(t)^2\Big)}_{=v(t)\frac{dv(t)}{dt} \text{ by the chain rule}}\Big]\,dt$$

$$= \underbrace{\frac{1}{2}m\,v(t_b)^2}_{\substack{=\text{Kinetic energy}\\ \text{at time } t_b}} - \frac{1}{2}m\,v(t_a)^2 = \text{change in Kinetic energy}$$

Rocket Equation:

exhaust: $R = \frac{dm(t)}{dt}$ in $\frac{kg}{s}$ at constant velocity v_e



$$\text{Force} = \text{momentum change} = \frac{d}{dt}\big(m(t)(-v_e)\big) = -R\,v_e = \text{mass} \times \text{acceleration} = m(t)\,\frac{dv(t)}{dt}$$

Suppose initial mass $= m_0$ and final mass $= m_f$, i.e., $m_0 - m_f$ kg of fuel are burnt.
(E.g., if R is constant, this takes time $T = \frac{m_0 - m_f}{R}$.)

$$\Rightarrow \text{gain in velocity } \Delta v = \int_0^T \frac{dv(t)}{dt}\,dt = \int_0^T \left(\frac{-R\,v_e}{m(t)}\right) dt$$

$$= -v_e \int_0^T \frac{1}{m(t)} \underbrace{\frac{dm(t)}{dt}\,dt}$$

substitution: $m(t) = m \Rightarrow m'(t)\,dt = dm$
with $m(T) = m_f$, $m(0) = m_0$

$$= -v_e \int_{m_0}^{m_f} \frac{1}{m}\,dm$$

$$= -v_e \ln m \Big|_{m_0}^{m_f}$$

$$= -v_e \left(\ln m_f - \ln m_0\right)$$

$$= v_e \ln \frac{m_0}{m_f}$$

$\Rightarrow$ Need $\Delta m = m_0 - m_f$ kg of fuel to increase velocity by $\boxed{\Delta v = v_e \ln \frac{m_0}{m_f}}$.