Calculus and Elements of Linear Algebra I — Session 18

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3. Integrals

Topic 3.E: Improper Integrals

Sometimes we can also integrate at singularities or ∞.

Definition:

- Let $f:[a,\infty)\to\mathbb{R}$ be integrable on $[a,r]$ for any $r>a$.

  Then $\boxed{\int_a^\infty f(x)\,dx := \lim_{r\to\infty}\int_a^r f(x)\,dx.}$ (Analogously: $\int_{-\infty}^a f(x)\,dx := \lim_{r\to-\infty}\int_r^a f(x)\,dx$.)

- Let $f:\mathbb{R}\to\mathbb{R}$ be integrable on any interval $[a,b]$.

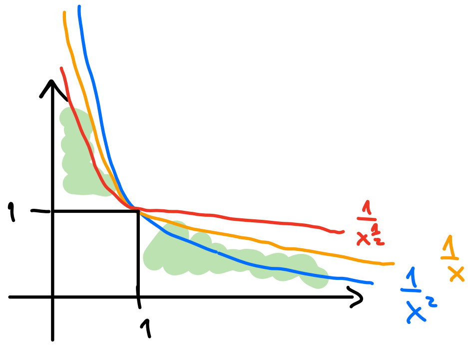
  Then $\boxed{\int_{-\infty}^\infty f(x)\,dx := \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx.}$ (Both limits need to exist!)

- Let $f:(a,b]\to\mathbb{R}$ be integrable on $[r,b]$ for any $a<r<b$, and have a vertical asymptote at a.

  Then $\boxed{\int_a^b f(x)\,dx := \lim_{r\to a}\int_r^b f(x)\,dx}.$

These integrals are called **improper integrals** if the limits exist.

Examples:



- Let $\alpha \neq 1$: $\int_1^\infty \frac{1}{x^\alpha}\,dx = \lim_{r\to\infty} \int_1^r \frac{1}{x^\alpha}\,dx = \lim_{r\to\infty} \frac{1}{1-\alpha} x^{-\alpha+1}\Big|_1^r = \frac{1}{1-\alpha} \lim_{r\to\infty} \left(r^{-\alpha+1} - 1\right)$

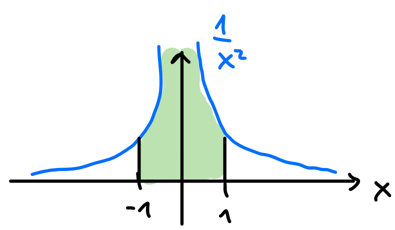
$$= \begin{cases} \frac{1}{\alpha-1} & \text{for } -\alpha+1<0 \text{, i.e., } \alpha>1, \\ \text{divergent} & \text{for } -\alpha+1>0 \text{, i.e., } \alpha<1. \end{cases}$$

At $\alpha = 1$: $\int_1^\infty \frac{1}{x}\,dx = \lim_{r\to\infty} \ln x\Big|_1^r = \lim_{r\to\infty} (\ln r - 0) \longrightarrow \infty$

- Let $\alpha \neq 1$: $\int_0^1 \frac{1}{x^\alpha}\,dx = \frac{1}{1-\alpha} \lim_{r\to 0} \left(1 - r^{-\alpha+1}\right) = \begin{cases} \text{divergent} & \text{for } \alpha>1, \\ \frac{1}{1-\alpha} & \text{for } \alpha<1. \end{cases}$

At $\alpha = 1$: $\int_0^1 \frac{1}{x}\,dx = \lim_{r\to 0} \ln x\Big|_r^1 = \lim_{r\to 0} (0 - \ln r) \longrightarrow \infty$

- $\cancel{\int_{-1}^1 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-1}^1 = -(1-(-1)) = -2}$???



$\frac{1}{x^2}$ has a vertical asymptote at $x=0$, so we need to split

$$\int_{-1}^1 \frac{1}{x^2}\,dx = \int_{-1}^0 \frac{1}{x^2}\,dx + \int_0^1 \frac{1}{x^2}\,dx = \lim_{r\to 0} \int_{-1}^r \frac{1}{x^2}\,dx + \lim_{r\to 0} \int_r^1 \frac{1}{x^2}\,dx = \lim_{r\to 0} \left[-\frac{1}{x}\right]_{-1}^r + \lim_{r\to 0} \left[-\frac{1}{x}\right]_r^1$$

$= \infty$

$\Rightarrow$ integral does not exist; area = infinite.

- $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{r\to\infty} \int_{-r}^{0} \frac{1}{1+x^2}\,dx + \lim_{r\to\infty} \int_{0}^{r} \frac{1}{1+x^2}\,dx$

$$= \lim_{r\to\infty} \arctan x \Big|_{-r}^{0} + \lim_{r\to\infty} \arctan x \Big|_{0}^{r}$$

$$= \lim_{r\to\infty} \left(-\arctan(-r)\right) + \lim_{r\to\infty} \arctan r$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2}$$

$$= \pi$$

- For which values of p does the integral $\int_{e}^{\infty} \frac{1}{x(\ln x)^p}\,dx$ converge?

(We already know: divergent for $p=0$ (and thus $p<0$), but does $p>0$ help?)

$$\int_{e}^{\infty} \frac{1}{x(\ln x)^p}\,dx = \int_{1}^{\infty} \frac{1}{u^p}\,du = \begin{cases} \frac{1}{1-p} u^{-p+1} & \text{for } p>1, \\ \infty & \text{for } p \le 1. \end{cases}$$

substitution: $u(x) = \ln x$

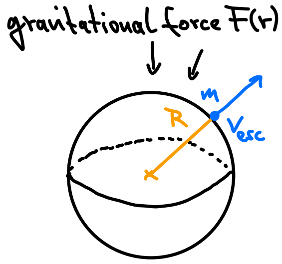
$\Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$

$u(e) = 1$, $u(\infty) = \infty$

$\Rightarrow$ Converges for $p>1$, diverges otherwise.

Application: escape velocity

planet with mass M, radius R



Goal: shoot object with mass m and initial velocity v_{esc} out of gravitational field of planet.

G = gravitational constant

Newton's law of gravity: $F(r) = \dfrac{GmM}{r^2}$.

r = distance from center of planet to center of object

We compute work W required to send object to ∞ and set it equal to the initial kinetic energy $K = \frac{1}{2} m v_{esc}^2$ (as computed in Session 17).

$$W = \int_R^\infty F(r)\,dr = GmM \int_R^\infty \frac{1}{r^2}\,dr = GmM\left[-r^{-1}\right]_R^\infty = \frac{GmM}{R}.$$

$W = K$ yields: $\dfrac{GmM}{R} = \frac{1}{2} m v_{esc}^2 \Rightarrow \boxed{v_{esc} = \sqrt{\dfrac{2GM}{R}}}$ (independent of m).